Sept 22+24 2014

F. Morrison

Pressure-driven flow in a tube

1

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

(Lectures 6+7 2014)

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

ρ constant

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde\tau_{xx}}{\partial x} + \frac{\partial \tilde\tau_{yx}}{\partial y} + \frac{\partial \tilde\tau_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde\tau_{xy}}{\partial x} + \frac{\partial \tilde\tau_{yy}}{\partial y} + \frac{\partial \tilde\tau_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde\tau_{xz}}{\partial x} + \frac{\partial \tilde\tau_{yz}}{\partial y} + \frac{\partial \tilde\tau_{zz}}{\partial z}\right) + \rho g_z$$

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tilde\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde\tau_{\theta r}}{\partial \theta} - \frac{\tilde\tau_{\theta\theta}}{r} + \frac{\partial \tilde\tau_{zr}}{\partial z}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde\tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde\tau_{z\theta}}{\partial z} + \frac{\tilde\tau_{\theta r} - \tilde\tau_{r\theta}}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tilde\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde\tau_{\theta z}}{\partial \theta} + \frac{\partial \tilde\tau_{zz}}{\partial z}\right) + \rho g_z$$

$$\boxed{\frac{\partial v_z}{\partial z} = 0}$$

unidir flow

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$

$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde\tau_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde\tau_{\theta r}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde\tau_{\phi r}}{\partial \phi} - \frac{\tilde\tau_{\theta\theta} + \tilde\tau_{\phi\phi}}{r}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$

$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde\tau_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde\tau_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde\tau_{\phi\theta}}{\partial \phi} + \frac{\tilde\tau_{\theta r} - \tilde\tau_{r\theta}}{r} - \frac{\tilde\tau_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$

$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde\tau_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\tilde\tau_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tilde\tau_{\phi\phi}}{\partial \phi} + \frac{\tilde\tau_{\phi r} - \tilde\tau_{r\phi}}{r} + \frac{\tilde\tau_{\phi\theta}\cot\theta}{r}\right) + \rho g_\phi$$

tube flow ②

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$
\begin{aligned}
\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) &= -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x \\
\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) &= -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \\
\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}
$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$
\begin{aligned}
\rho\left(\cancel{\frac{\partial v_r}{\partial t}}+\cancel{v_r\frac{\partial v_r}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}-\cancel{\frac{v_\theta^2}{r}}+\cancel{v_z\frac{\partial v_r}{\partial z}}\right) &= -\frac{\partial P}{\partial r}+\mu\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right)}+\cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}-\cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}+\cancel{\frac{\partial^2 v_r}{\partial z^2}}\right)+\cancel{\rho g_r} \\
\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}}+\cancel{v_r\frac{\partial v_\theta}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}}+\cancel{\frac{v_r v_\theta}{r}}+\cancel{v_z\frac{\partial v_\theta}{\partial z}}\right) &= -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r}\right)}+\cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}}+\cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}+\cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right)+\cancel{\rho g_\theta} \\
\rho\left(\frac{\partial v_z}{\partial t}+\cancel{v_r\frac{\partial v_z}{\partial r}}+\cancel{\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}}+v_z\frac{\partial v_z}{\partial z}\right) &= -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z
\end{aligned}
$$

$g_z = g$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$v_r = 0$

$v_\theta = 0$

steady flow

θ symmetry

$\frac{\partial v_z}{\partial z} = 0$

$$
\begin{aligned}
&\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r}\right) \\
&\quad = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right. \\
&\qquad\qquad \left.-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_r
\end{aligned}
$$

$$
\begin{aligned}
&\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^2\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right. \\
&\qquad\qquad \left.+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right)+\rho g_\theta
\end{aligned}
$$

$$
\begin{aligned}
&\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r v_\phi}{r}+\frac{v_\phi v_\theta\cot\theta}{r}\right) \\
&\quad = -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right. \\
&\qquad\qquad \left.+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right)+\rho g_\phi
\end{aligned}
$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.

2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

$$\underline{g} = \begin{pmatrix} g_r \\ g_\theta \\ g_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}_{r\theta z}$$

r-component:

$$0 = -\frac{\partial P}{\partial r} \qquad \boxed{\frac{\partial P}{\partial r} = 0}$$

θ-component

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} \Rightarrow \boxed{\frac{\partial P}{\partial \theta} = 0}$$

We conclude:

$\Rightarrow P = P(z)$

z-component

$$0 = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right) + \rho g$$

5

$$\frac{dp^{(z)}}{dz} = \mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{d\overset{v_z(r)}{\overset{\downarrow}{V_z}}}{dr}\right)\right) + \rho g$$

$$\underbrace{\frac{dp}{dz} - \rho g}_{f(z)} = \underbrace{\mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right)\right)}_{\tilde{f}(r)} = \lambda$$

__LHS__: (left hand side)

$$\frac{dp}{dz} - \rho g = \lambda$$

6

$$\frac{dP}{dz} = (\lambda + \rho g)$$

$$P = (\lambda + \rho g) z + c_1$$

two constants (unknown)

Aside

$$\frac{dy}{dx} = 3$$

$$y = 3x + c_1$$

Bounding Conditions on Pressure:

$z = 0 \quad P = P_0$

$z = L \quad P = P_L$

$$P_0 = (\lambda + \rho g)(0) + c_1$$

$$\boxed{c_1 = P_0}$$

$$P_L = (\lambda + \rho g) L + P_0$$

$$\boxed{\lambda = \frac{P_L - P_0}{L} - \rho g}$$

RHS: (right-hand side)

$$\lambda = \mu \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dV_z}{dr} \right) \right)$$

$$\frac{\lambda}{\mu} r = \frac{d}{dr} \underbrace{\left(r \frac{dV_z}{dr} \right)}_{\equiv \Psi}$$

$$\frac{d\Psi}{dr} = \left(\frac{\lambda}{\mu} \right) r$$

ASIDE

$$\frac{dy}{dx} = 7x$$

$$y = \frac{7x^2}{2} + C$$

$$\Psi = \left(\frac{\lambda}{\mu}\right)\frac{r^2}{2} + c_2$$

8

$\Psi = \uparrow$ from above

$$\Psi = \left(\frac{\lambda}{2\mu}\right) r^2 + c_2 = \underbrace{r\frac{dV_z}{dr}}$$

$$\frac{dV_z}{dr} = \left(\frac{\lambda}{2\mu}\right) r + \frac{c_2}{r}$$

ASIDE

$$\int \frac{dx}{x} = \ln x + c$$

$$\int r\,dr = \frac{r^2}{2}$$

$$\boxed{V_z = \left(\frac{\lambda}{2\mu}\right)\frac{r^2}{2} + c_2 \ln r + c_3}$$

now we need boundary conditions ...

Boundary Conditions:

BC1: $r = R$ $\quad v_z = 0$ no slip at wall

BC2: $r = 0$ $\quad \frac{dv_z}{dr} = 0$ maximum at center (symmetric flow)

substitute 2nd BC:

$$\frac{dv_z}{dr} = \left(\frac{\lambda}{2\mu}\right) r + \frac{c_2}{r}$$

at $r=0$ this is $-\infty$!!

$$0 = 0 + \frac{c_2}{0} \implies \boxed{c_2 = 0}$$

??

if $c_2 = 0$ then

$$v_z = \left(\frac{\lambda}{4\mu}\right) r^2 + c_3$$

$r = R$ $v_z = 0 \Rightarrow 0 = \frac{\lambda R^2}{4\mu} + c_3$

$$\boxed{c_3 = -\frac{\lambda R^2}{4\mu}}$$

$$v_z = \frac{\lambda}{4\mu} r^2 - \frac{\lambda}{4\mu} R^2$$

$$v_z = \frac{\lambda}{4\mu}\left(r^2 - R^2\right)$$

$$\boxed{v_z = \left(\frac{P_L - P_0}{L} - \rho g\right)\frac{1}{4\mu}\left(r^2 - R^2\right)}$$

One way to check algebra is to verify units:

$$\frac{\cancel{Pa}}{1}\ \frac{1}{\cancel{m}}\ \frac{1}{\cancel{Pa}\,s}\ m^{\cancel{2}}$$

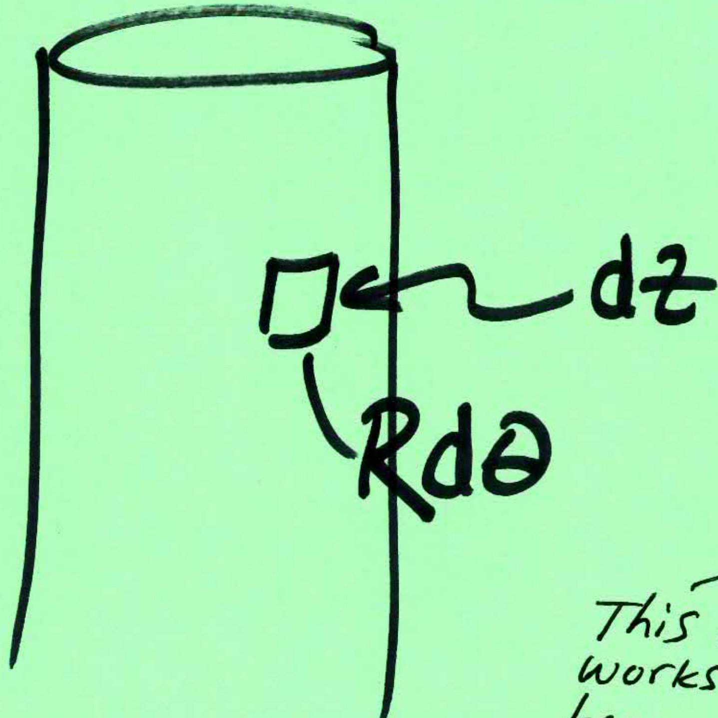
$$[=]\ \frac{m}{s}\ \checkmark$$

Another way is to back-substitute the BC.

what is force on wall?

$$F = \int_0^L \int_0^{2\pi} \overbrace{\tau_{rz}\Big|_{r=R}}^{\text{force/area}} \; \overbrace{R\,d\theta\,dz}^{\text{area}}$$

form dA:



Newton's Law of Viscosity:

$$\boxed{\tau_{rz} = \mu \frac{dV_z}{dr}}$$

This only works because it's unidirectional flow.

$$\frac{dV_z}{dr} = \left(\frac{P_L - P_0 - \rho g L}{L}\right)(\cancel{2}r)\left(\frac{1}{\cancel{4}_{2}\mu}\right)$$

$$\tau_{rz} = \mu \frac{dV_z}{dr} = \left(\frac{P_L - P_0 - \rho g L}{L}\right)\frac{r}{2}$$

$$\tau_{rz}\Big|_{r=R} = \left(\frac{P_L - P_0 - \rho g L}{2L}\right)R$$

$$F = \int_0^L \int_0^{2\pi} \underbrace{\tau_{rz}\Big|_{r=R}}_{\frac{(P_L - P_0 - \rho g L)R}{2L}} R\, d\theta\, dz$$

$$= \text{const} = \tau_R$$

$$F = \tau_R R \underbrace{\int_0^L \int_0^{2\pi} d\theta\, dz}_{2\pi L}$$

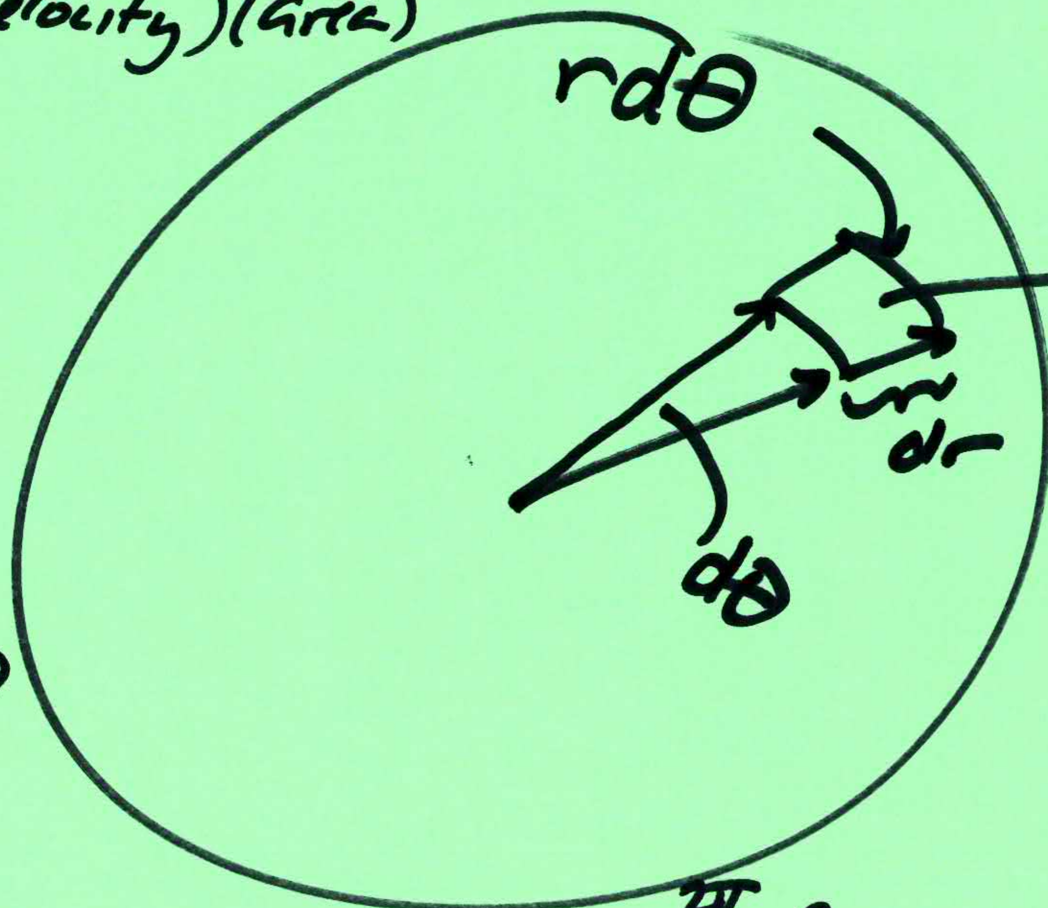
$$\boxed{F = \tau_R 2\pi R L}$$

How do we calculate Q?

$Q = \iint (\text{velocity})(\text{area})$

area?

cross section of tube



$dA = r\,dr\,d\theta$

$$Q = \int_0^{2\pi}\int_0^{R} v_z \, r\,dr\,d\theta$$