

2016 - Tube Flow

0.25

EXAMPLE II:
 Pressure-driven flow
 of a Newtonian fluid
 in a tube:
(Poiseuille flow)

- steady state
- constant ρ
- well developed
- long tube
- pressure p_o at top
- pressure p_L at bottom

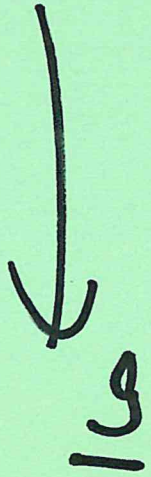
fluid \downarrow \underline{g}

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}$$

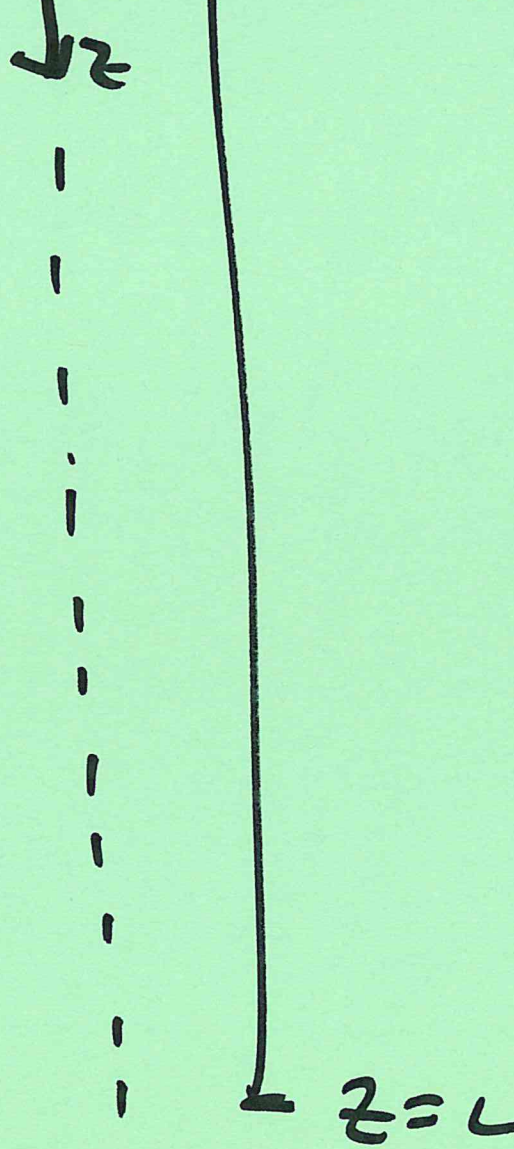
roz

Flow
in
a tube

— pressure —
driven



r — $0 = z$



BC on P:

$$z=0$$

$$P=P_0$$

$$z=L$$

$$P=P_L$$

$$v = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}$$

①

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

→ Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\rho = \text{constant} \quad \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Pressure driven flow in a tube

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial(r \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{rz}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial(r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{r\theta}}{\partial r} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial(r \tilde{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} + \frac{v_\phi^2}{r} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial(r^2 \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{r\phi}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

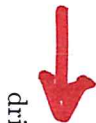
$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r^2} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial(r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\theta r}}{\partial r} - \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r^3} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial(r^3 \tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi r}}{\partial r} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta} \cot \theta}{r} \right) + \rho g_\phi$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

Axisymmetry



Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2}{r} - \frac{v_\phi^2}{r \sin^2 \theta} \right) &= -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) - \frac{2}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + v_r v_\theta - \frac{v_\phi^2 \cot \theta}{r} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \\ &\quad + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \rho g_\theta \\ \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right) \\ &\quad + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \theta} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \rho g_\phi \end{aligned}$$

Stokes Stokes

continuity

g = (0, 0, g)

Note: the r -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \mathbf{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

r-component N-S:

$$0 = -\frac{\partial P}{\partial r} \Rightarrow \text{Pressure is not a function of } r$$

θ -component N-S:

$$0 = -\frac{1}{r} \frac{\partial P}{\partial \theta}$$

$$0 = \frac{\partial P}{\partial \theta} \Rightarrow \text{Pressure is not a function of } \theta$$

\Rightarrow Pressure is at most a function of z .

(3)

z - component of NS:

(7)

$$0 = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right) + \rho g$$

$P = P(z)$ only \Rightarrow $\frac{dP}{dz}$

$\frac{dv_z}{dr}$

$$\underbrace{\frac{dP}{dz} - \rho g}_{\text{is only a function of } z} = \underbrace{\mu \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) \right)}_{\text{is only a function of } r} = \underbrace{\lambda}_{\text{both = same const. } \lambda}$$

(separation of variables)

LHS: (left hand side, pressure(z)) (5)

$$\frac{dP}{dz} - \rho g = \lambda$$

$$\frac{dP}{dz} = \lambda + \rho g$$

$$\int dP = \int (\lambda + \rho g) dz$$

$$P = (\lambda + \rho g)z + C_1$$

Given

$$\left\{ \begin{array}{l} \text{BC: } z=0 \quad P=P_0 \quad \Rightarrow \quad C_1 = P_0 \\ z=L \quad P=P_L \quad \Rightarrow \quad P_L = (\lambda + \rho g)L + P_0 \end{array} \right.$$

(algebra
missing)

$$\lambda = \frac{P_L - P_0}{L} - \rho g$$

$$P = \left(\frac{P_L - P_0}{L} \right) z + P_0$$

Final Pressure profile
in z -direction

RHS (right-hand side, $V_2(r)$)

(7)

→ λ is determined from boundary conditions.

$$\lambda = \mu \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{dV_2}{dr} \right) \right)$$
$$\frac{\lambda}{\mu} r = \frac{d}{dr} \left(r \frac{dV_2}{dr} \right)$$

$\equiv \Phi$

(don't differentiate when integrating, if you can help it!)

$$\frac{d\Phi}{dr} = \left(\frac{\lambda}{\mu} \right) r$$

integrate:

$$\Phi = \frac{\lambda}{\mu} \frac{r^2}{2} + C_2$$

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$$\Phi = r \frac{dv_z}{dr} = \left(\frac{\lambda}{2\mu}\right) r^2 + C_2$$

$\frac{dv_z}{dr} = 0$
at $r=0 \Rightarrow$
 $C_2 = 0$

Applying
BC1: at $r=0$
 $\frac{dv_z}{dr} = 0$

~~$$\frac{dv_z}{dr} = \frac{\lambda}{2\mu} r + \frac{C_2}{r} \Rightarrow C_2$$~~

undefined!

~~$$v_z = \left(\frac{\lambda}{2\mu}\right) \frac{r^2}{2} + C_2 \ln r + C_3$$~~

(yikes!)

BC:

$r=0$	$\frac{dv_z}{dr} = 0$
$r=R$	$v_z = 0$

(the maximum is at the center)

$$v_z = \frac{\lambda}{4\mu} r^2 + C_3$$

Bc 2: $0 = \left(\frac{\lambda}{4\mu}\right) R^2 + C_3$

$$\Rightarrow C_3 = -\left(\frac{\lambda}{4\mu}\right) R^2$$

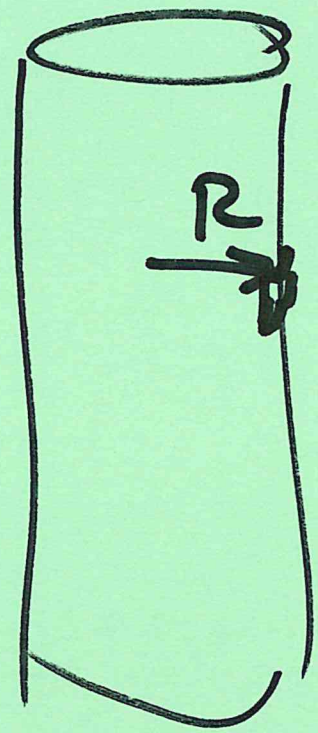
$$v_z = \frac{\lambda}{4\mu} (r^2 - R^2)$$

$$v_z = \left(\frac{P_L - P_0}{4\mu L} - \frac{\rho g}{4\mu} \right) (r^2 - R^2)$$

What is total
Force on wall?

$$F_{\text{wall}} = \int_0^L \int_0^{2\pi} \tau_{rz} \Big|_{r=R} (R \, d\theta \, dz)$$

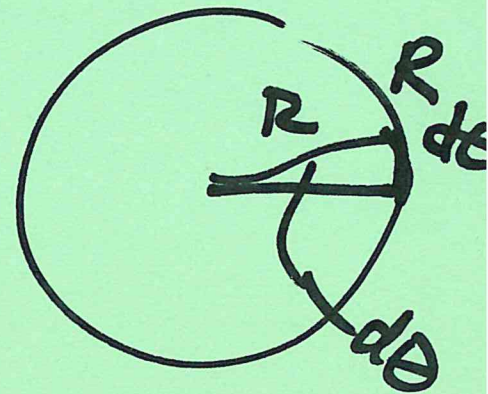
$\left(\frac{\text{force}}{\text{area}} \right) (\text{area})$



What is $\tau_{rz}|_{r=R}$?

Newton's Law of Visc.

$$\tau_{rz} = \mu \frac{dv_z}{dr}$$



Engineering Quantities of Interest

(any flow)

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component of force on the wall

$$F_z = \hat{e}_z \cdot \iint_S [\hat{n} \cdot (-p\underline{I} + \underline{\tilde{\tau}})]_{surface} dS$$

How would we get it from this? (11)



For more complex flows, we use the **Gibbs notation** versions (will discuss soon).

tension on
a surface

$$\left(\hat{n} \cdot \underline{\underline{\underline{\tau}}}} \right)$$

at the
surface

dS

surface
element
at the
surface

unit
normal
to the surface

"SLASH + BURN"

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The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof. Faith A. Morrison, Michigan Technological University

Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$\nabla \cdot \mathbf{v} = 0$
 $u_r = 0$
 $\partial v_{\theta} = 0$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tilde{\tau}_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tilde{\tau}_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} & 2 \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \end{pmatrix}_{r\theta\phi}$$

$\nabla \cdot \mathbf{v} = 0$
confinity

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

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$$\frac{2}{1} = \mu \begin{pmatrix} 0 & 0 & \frac{dV_z}{dr} \\ 0 & 0 & 0 \\ \frac{dV_z}{dr} & 0 & 0 \end{pmatrix} \text{ } z_{\theta z}$$

$$\tau_{zr} = \mu \frac{dV_z}{dr}$$

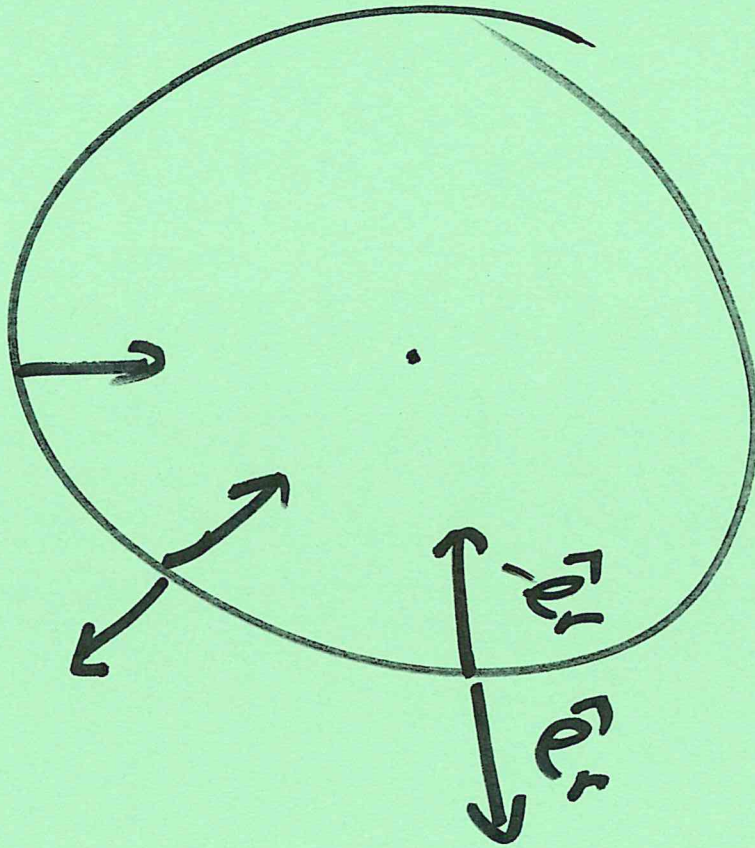
$$\tau_{rz} = \mu \frac{dV_z}{dr}$$

A
SAME

$$-P \underline{\underline{I}} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & P \end{pmatrix} \quad r \theta z$$

$$\underline{\underline{L}} + P \underline{\underline{I}} = \begin{pmatrix} -P & 0 & \mu \frac{dv_z}{dr} \\ 0 & -P & 0 \\ \mu \frac{dv_z}{dr} & 0 & -P \end{pmatrix} \quad r \theta z$$

What is \hat{n} ?



$$\hat{n} = \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$(\ddot{n}, \ddot{\pi}) =$$

$$(-1 \ 0 \ 0)_{\text{rot}}$$

$$\begin{pmatrix} -P & 0 & \mu \frac{dv_t}{dr} \\ 0 & -P & 0 \\ \mu \frac{dv_t}{dr} & 0 & -P \end{pmatrix}_{\text{rot}}$$

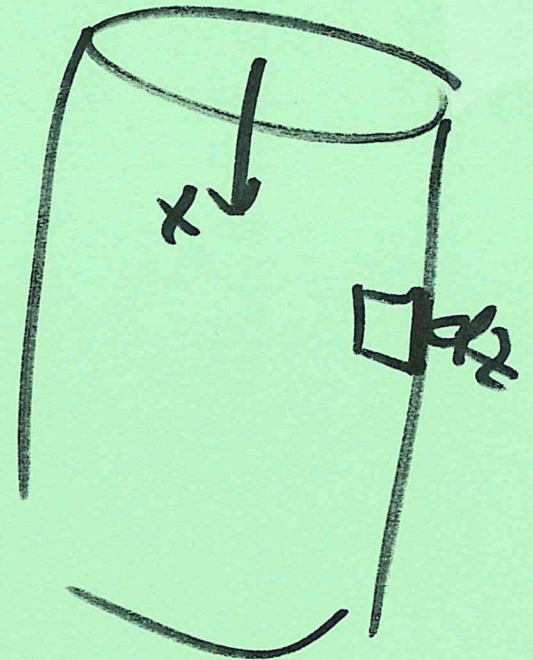
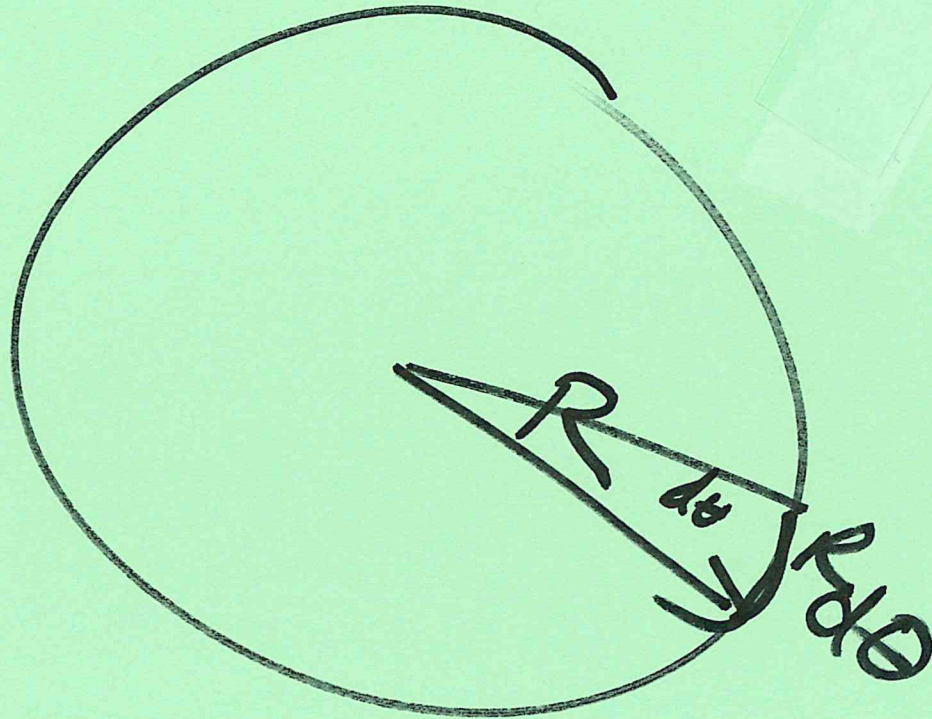
$$= (P \ 0 \ -\mu \frac{dv_t}{dr})_{\text{rot}}$$

$$\hat{n} \cdot \hat{\Pi} = \left(P \quad 0 \quad -\mu \frac{dU_z}{dr} \right)$$

$[\hat{n} \cdot \hat{\Pi}]$ at surface
~~~~~?

$$(\hat{n} \cdot \hat{\Pi})|_{r=R}$$

↖ location of surface



$$ds = R d\theta dz$$

at surface?  $r = R$

assembly:

$$F_{wall} = \iint (\hat{n} \cdot \underline{\underline{\pi}}) \text{ at surface } ds$$

$$F_{wall} = \int_0^L \int_0^{2\pi} \left( \begin{matrix} p \\ 0 \\ -\mu \frac{dv_z}{dr} \end{matrix} \right) r d\theta dz \Big|_{r=R}$$

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