

Equations Summary from Inside Cover of Morrison, 2013

Mechanical Energy Balance $\frac{\Delta p}{\rho} + \frac{\Delta \langle v \rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by fluid}}{m}$ $\begin{cases} \alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1 \end{cases}$

$$F_{friction} = \left[4f \frac{L}{D} + \sum_{fittings_i} n_i K_{f,i} \right] \frac{\langle v \rangle^2}{2}$$

Fanning Friction Factor (pipe flow)

$$f = \frac{F_{drag}}{\frac{1}{2} \rho \langle v \rangle^2 (2\pi RL)} = \frac{\Delta p D}{2L \rho \langle v \rangle^2}$$

Note this is correct; there is an error on the inside cover

Drag Coefficient (sphere drop)

$$C_D = \frac{F_{drag}}{\frac{1}{2} \rho v_{\infty}^2 (\pi R^2)} = \frac{4gD(\rho_{body} - \rho)}{3\rho v_{\infty}^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\mathbf{P}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v}) \rho \underline{v} dS = \sum_{on CV} \underline{f}$$

Hydrostatic Pressure

$$p_{bottom} = p_{top} + \rho gh$$

Hagen-Poiseuille Equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Prandtl Equation (steady, turbulent tube flow)

$$\frac{1}{\sqrt{f}} = -4.0 \log \left(\frac{4.67}{Re\sqrt{f}} \right) + 2.28$$

Stokes-Einstein-Sutherland Equation (steady, slow flow around a sphere)

$$F_{drag} = 6\pi R\mu v_{\infty}$$

Macroscopic Momentum Balance on a CV

$$\frac{d\mathbf{P}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos(\theta) \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$
 $\begin{cases} \beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1 \end{cases}$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla \cdot \underline{v} = 0$$

Total stress tensor $\underline{\tilde{\Pi}} = -p\underline{I} + \underline{\tilde{\tau}}$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure $\mathcal{P} \equiv p + \rho gh$

Newtonian constitutive equation $\underline{\tilde{\tau}} = \mu \left(\nabla \underline{v} + (\nabla \underline{v})^T \right)$

$$= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

Total molecular fluid force on a finite surface \mathcal{S} $\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\tilde{\Pi}} \right]_{\text{at surface}} dS$

Stationary fluid $\left[\hat{n} \cdot \underline{\tilde{\Pi}} \right] = -p\hat{n}$

Moving fluid $\left[\hat{n} \cdot \underline{\tilde{\Pi}} \right] = -p\hat{n} + \hat{n} \cdot \underline{\tilde{\tau}}$

Total fluid torque on a finite surface \mathcal{S} $\underline{\mathcal{T}} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\tilde{\Pi}} \right) \right]_{\text{at surface}} dS$

Total flow rate out through a finite surface \mathcal{S} $Q = \dot{V} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{v} \right]_{\text{at surface}} dS$

Average velocity across a finite surface \mathcal{S} $\langle v \rangle = \frac{Q}{\mathcal{S}}$

Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dx dy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dx dz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dy dz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = r dr d\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = R d\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin \theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dx dy dz$
cylindrical	$dV = r dr d\theta dz$
spherical	$dV = r^2 \sin \theta dr d\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r \sin \theta \cos \phi$	$\hat{e}_r = (\sin \theta \cos \phi \hat{e}_x) + (\sin \theta \sin \phi \hat{e}_y) + \cos \theta \hat{e}_z$
	$y = r \sin \theta \sin \phi$	$\hat{e}_\theta = (\cos \theta \cos \phi) \hat{e}_x + (\cos \theta \sin \phi) \hat{e}_y + (-\sin \theta) \hat{e}_z$
	$z = r \cos \theta$	$\hat{e}_\phi = (-\sin \phi) \hat{e}_x + \cos \phi \hat{e}_y$
cylindrical	$x = r \cos \theta$	$\hat{e}_r = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y$
	$y = r \sin \theta$	$\hat{e}_\theta = (-\sin \theta) \hat{e}_x + \cos \theta \hat{e}_y$
	$z = z$	$\hat{e}_z = \hat{e}_z$

$$\text{Divergence Theorem} \quad \iint_S \hat{n} \cdot \underline{F} dS = \iiint_V \nabla \cdot \underline{F} dV$$

$$\text{Stokes Theorem} \quad \oint_C \hat{t} \cdot \underline{F} dl = \iint_S \hat{n} \cdot (\nabla \times \underline{F}) dS$$

Vector identities:

$$\nabla \cdot \nabla \times \underline{F} = 0 \quad (\text{Divergence of curl} = 0)$$

$$\nabla \times \nabla f = 0 \quad (\text{Curl of gradient} = 0)$$

$$\nabla (fg) = f \nabla g + g \nabla f$$

$$\underline{F} \cdot \nabla \underline{F} = \frac{1}{2} \nabla (\underline{F}^2) - \underline{F} \times (\nabla \times \underline{F})$$

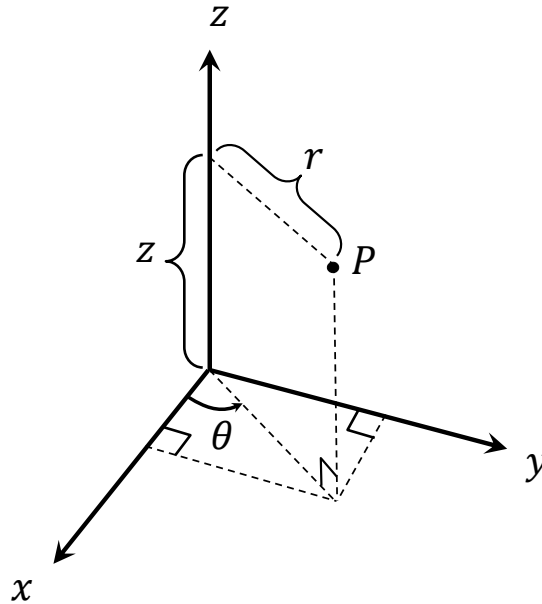
$$\nabla \cdot (f \underline{F}) = f \nabla \cdot \underline{F} + \underline{F} \cdot \nabla f$$

$$\nabla \times \nabla \times \underline{F} = \nabla (\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$$

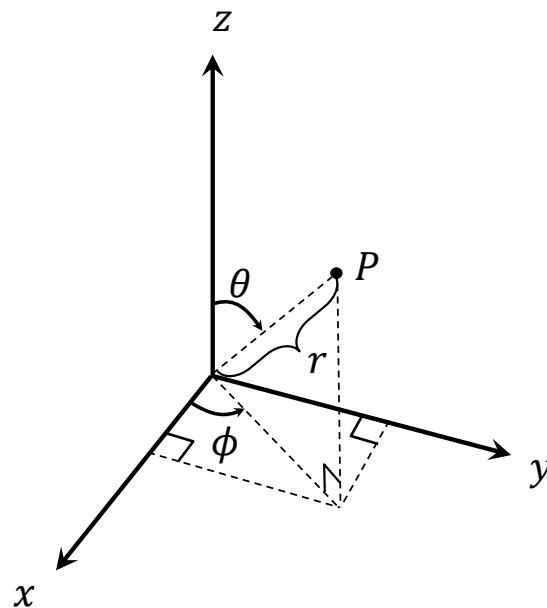
$$\nabla \cdot (\underline{F} \times \underline{G}) = \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})$$

The equations in F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge, 2013) assume the following definitions of the cylindrical and spherical coordinate systems.

Cylindrical Coordinate System: Note that the θ -coordinate swings around the z -axis



Spherical Coordinate System: Note that the θ -coordinate swings down from the z -axis; this is different from its definition in the cylindrical system above.



The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

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Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial(r \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2} \frac{\partial(r^2 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial(r \tilde{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\begin{aligned} &\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial(r^2 \tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{\theta r} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r \\ &\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial(r^3 \tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta \\ &\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) \\ &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial(r^3 \tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\tilde{\tau}_{\theta\phi} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\theta\theta} \cot \theta}{r} \right) + \rho g_\phi \end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned}\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z\end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned}\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z\end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\begin{aligned}\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) &= -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right. \\ &\quad \left. - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right. \\ &\quad \left. + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta \\ \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right) &= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right. \\ &\quad \left. + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi\end{aligned}$$

Note: the r -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

FACTORS FOR UNIT CONVERSIONS

Quantity	Equivalent Values
Mass	$1 \text{ kg} = 1000 \text{ g} = 0.001 \text{ metric ton} = 2.20462 \text{ lb}_m = 35.27392 \text{ oz}$ $1 \text{ lb}_m = 16 \text{ oz} = 5 \times 10^{-4} \text{ ton} = 453.593 \text{ g} = 0.453593 \text{ kg}$
Length	$1 \text{ m} = 100 \text{ cm} = 1000 \text{ mm} = 10^6 \text{ microns } (\mu\text{m}) = 10^{10} \text{ angstroms } (\text{\AA})$ $= 39.37 \text{ in} = 3.2808 \text{ ft} = 1.0936 \text{ yd} = 0.0006214 \text{ mile}$ $1 \text{ ft} = 12 \text{ in.} = 1/3 \text{ yd} = 0.3048 \text{ m} = 30.48 \text{ cm}$
Volume	$1 \text{ m}^3 = 1000 \text{ liters} = 10^6 \text{ cm}^3 = 10^6 \text{ ml}$ $= 35.3145 \text{ ft}^3 = 220.83 \text{ imperial gallons} = 264.17 \text{ gal}$ $= 1056.68 \text{ qt}$ $1 \text{ ft}^3 = 1728 \text{ in}^3 = 7.4805 \text{ gal} = 0.028317 \text{ m}^3 = 28.317 \text{ liters}$ $= 28\,317 \text{ cm}^3$
Force	$1 \text{ N} = 1 \text{ kg}\cdot\text{m}/\text{s}^2 = 10^5 \text{ dynes} = 10^5 \text{ g}\cdot\text{cm}/\text{s}^2 = 0.22481 \text{ lb}_f$ $1 \text{ lb}_f = 32.174 \text{ lb}_m\cdot\text{ft}/\text{s}^2 = 4.4482 \text{ N} = 4.4482 \times 10^5 \text{ dynes}$
Pressure	$1 \text{ atm} = 1.01325 \times 10^5 \text{ N}/\text{m}^2 \text{ (Pa)} = 101.325 \text{ kPa} = 1.01325 \text{ bars}$ $= 1.01325 \times 10^6 \text{ dynes}/\text{cm}^2$ $= 760 \text{ mm Hg at } 0^\circ \text{ C (torr)} = 10.333 \text{ m H}_2\text{O at } 4^\circ \text{ C}$ $= 14.696 \text{ lb}_f/\text{in}^2 \text{ (psi)} = 33.9 \text{ ft H}_2\text{O at } 4^\circ \text{ C}$ $100 \text{ kPa} = 1 \text{ bar}$
Energy	$1 \text{ J} = 1 \text{ N}\cdot\text{m} = 10^7 \text{ ergs} = 10^7 \text{ dyne}\cdot\text{cm}$ $= 2.778 \times 10^{-7} \text{ kW}\cdot\text{h} = 0.23901 \text{ cal}$ $= 0.7376 \text{ ft}\cdot\text{lb}_f = 9.47817 \times 10^{-4} \text{ Btu}$
Power	$1 \text{ W} = 1 \text{ J}/\text{s} = 0.23885 \text{ cal}/\text{s} = 0.7376 \text{ ft}\cdot\text{lb}_f/\text{s} = 9.47817 \times 10^{-4} \text{ Btu}/\text{s} = 3.4121 \text{ Btu}/\text{h}$ $= 1.341 \times 10^{-3} \text{ hp (horsepower)}$
Viscosity	$1 \text{ Pa}\cdot\text{s} = 1 \text{ N}\cdot\text{s}/\text{m}^2 = 1 \text{ kg}/\text{m}\cdot\text{s}$ $= 10 \text{ poise} = 10 \text{ dynes}\cdot\text{s}/\text{cm}^2 = 10 \text{ g}/\text{cm}\cdot\text{s}$ $= 10^3 \text{ cp (centipoise)}$ $= 0.67197 \text{ lb}_m/\text{ft}\cdot\text{s} = 2419.088 \text{ lb}_m/\text{ft}\cdot\text{h}$
Density	$1 \text{ kg}/\text{m}^3 = 10^{-3} \text{ g}/\text{cm}^3$ $= 0.06243 \text{ lb}_m/\text{ft}^3$ $10^3 \text{ kg}/\text{m}^3 = 1 \text{ g}/\text{cm}^3 = 62.428 \text{ lb}_m/\text{ft}^3$
Volumetric Flow	$1 \text{ m}^3/\text{s} = 35.3145 \text{ ft}^3/\text{s} = 15,850.2 \text{ gal}/\text{min (gpm)}$ $1 \text{ gpm} = 6.30907 \times 10^{-5} \text{ m}^3/\text{s} = 2.22802 \times 10^{-3} \text{ ft}^3/\text{s} = 3.7854 \text{ liter}/\text{min}$ $1 \text{ liter}/\text{min} = 0.26417 \text{ gpm}$

Temperature	$T(^{\circ}C) = \frac{5}{9}[T(^{\circ}F) - 32]$ $T(^{\circ}F) = \frac{9}{5}T(^{\circ}C) + 32 = 1.8T(^{\circ}C) + 32$
Absolute Temperature	$T(K) = T(^{\circ}C) + 273.15$ $T(^{\circ}R) = T(^{\circ}F) + 459.67$
Temperature Interval (ΔT)	$1 C^{\circ} = 1 K = 1.8 F^{\circ} = 1.8 R^{\circ}$ $1 F^{\circ} = 1 R^{\circ} = (5/9) C^{\circ} = (5/9) K$

USEFUL QUANTITIES

$$SG = \rho(20^{\circ}C) / \rho_{\text{water}}(4^{\circ}C)$$

$$\rho_{\text{water}}(4^{\circ}C) = 1000 \text{ kg/m}^3 = 62.43 \text{ lb}_m/\text{ft}^3 = 1.000 \text{ g/cm}^3$$

$$\rho_{\text{water}}(25^{\circ}C) = 997.08 \text{ kg/m}^3 = 62.25 \text{ lb}_m/\text{ft}^3 = 0.99709 \text{ g/cm}^3$$

$$g = 9.8066 \text{ m/s}^2 = 980.66 \text{ cm/s}^2 = 32.174 \text{ ft/s}^2$$

$$\begin{aligned} \mu_{\text{water}}(25^{\circ}C) &= 8.937 \times 10^{-4} \text{ Pa}\cdot\text{s} = 8.937 \times 10^{-4} \text{ kg/m}\cdot\text{s} \\ &= 0.8937 \text{ cp} = 0.8937 \times 10^{-2} \text{ g/cm}\cdot\text{s} = 6.005 \times 10^{-4} \text{ lb}_m/\text{ft}\cdot\text{s} \end{aligned}$$

Composition of air:	N ₂	78.03%
	O ₂	20.99%
	Ar	0.94%
	CO ₂	0.03%
	H ₂ , He, Ne, Kr, Xe	<u>0.01%</u>
		100.00%

$$M_{\text{air}} = 29 \text{ g/mol} = 29 \text{ kg/kmol} = 29 \text{ lb}_m/\text{lbmole}$$

$$\hat{C}_{p,\text{water}}(25^{\circ}C) = 4.182 \text{ kJ/kg}\cdot\text{K} = 0.9989 \text{ cal/g}\cdot\text{C} = 0.9997 \text{ Btu/lb}_m\cdot\text{F}$$

$$\begin{aligned} R &= 8.314 \text{ m}^3\text{Pa/mol}\cdot\text{K} = 0.08314 \text{ liter}\cdot\text{bar/mol}\cdot\text{K} = 0.08206 \text{ liter}\cdot\text{atm/mol}\cdot\text{K} \\ &= 62.36 \text{ liter}\cdot\text{mm Hg/mol}\cdot\text{K} = 0.7302 \text{ ft}^3\cdot\text{atm/lbmole}\cdot\text{R} \\ &= 10.73 \text{ ft}^3\cdot\text{psia/lbmole}\cdot\text{R} \\ &= 8.314 \text{ J/mol}\cdot\text{K} \\ &= 1.987 \text{ cal/mol}\cdot\text{K} = 1.987 \text{ Btu/lbmole}\cdot\text{R} \end{aligned}$$

Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison

I. Data Correlations for Flow through Smooth Pipes

A. All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond $Re = 4000$ this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in f ; when flow rate is known, Δp may be found directly; when Δp is known, Q or (v) must be solved for iteratively.

$$\text{Morrison (2013)} \quad f = \left(\frac{0.0076 \left(\frac{3170}{Re} \right)^{0.165}}{1 + \left(\frac{3170}{Re} \right)^{7.0}} \right) + \frac{16}{Re} \quad (1)$$

B. $4,000 \leq Re \leq 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and must be solved iteratively except when f is known (Morrison, equation 7.156). This is good only for $Re > 4,000$ /

$$\text{Prandtl or VonKarman-Nikuradse (Denn, 1980)} \quad \frac{1}{\sqrt{f}} = 4.0 \log(Re\sqrt{f}) - 0.40 \quad (2)$$

C. $4,000 \leq Re \leq 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for $Re > 4,000$.

$$\text{Simplified Turbulent (White, 1974)} \quad f = \frac{1.02}{4} (\log Re)^{-2.5} \quad (3)$$

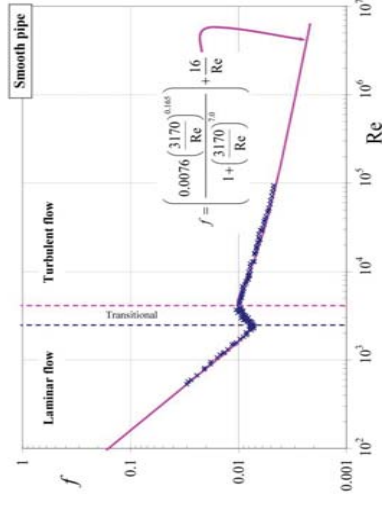


Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond $Re = 10^6$ is not recommended; for $Re > 4000$ equation 3 follows the Prandtl equation. (Morrison, 2013, p532)

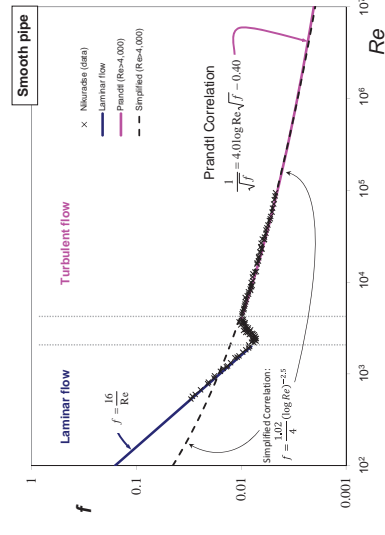


Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with. (Morrison, 2013, p531)

II. Data Correlations for Flow Around a Sphere

A. All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

$$\text{Morrison (2013)} \quad C_D = \frac{24}{Re} + \frac{2.6 \left(\frac{Re}{5.0}\right)^{1.32}}{1 + \left(\frac{Re}{5.0}\right)^{1.32}} + \frac{0.411 \left(\frac{Re}{263,000}\right)^{-7.94}}{1 + \left(\frac{Re}{263,000}\right)^{-7.94}} + \frac{0.25 \left(\frac{Re}{10^6}\right)}{1 + \left(\frac{Re}{10^6}\right)} \quad (4)$$

Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$$Re < 2 \quad C_D = \frac{24}{Re} \quad (5)$$

$$0.1 \leq Re \leq 1,000 \quad C_D = \frac{24}{Re} (1 + 0.14Re^{0.7}) \quad (6)$$

$$1,000 \leq Re \leq 2.6 \times 10^5 \quad C_D = 0.445 \quad (7)$$

$$2.8 \times 10^5 \leq Re \leq 10^6 \quad \log \frac{C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3 \quad (8)$$

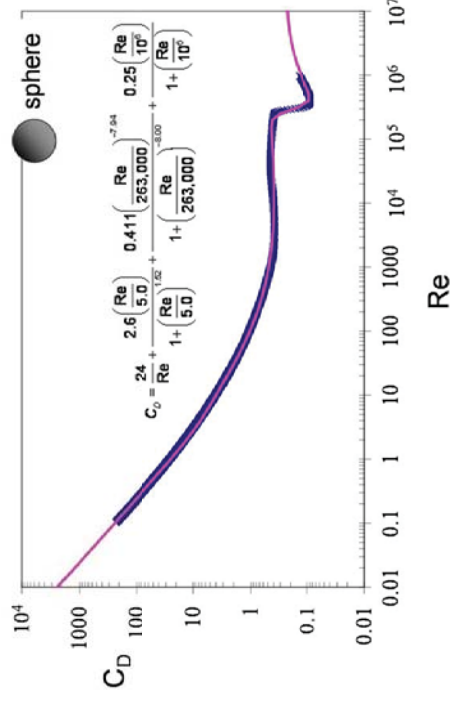


Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended. (Morrison, 2013, p625)

References

- M. Denn, *Process Fluid Mechanics* (Prentice Hall, Englewood Cliffs, NJ, 1980)
- F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
- F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof. Faith A. Morrison, Michigan Technological University

Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tau_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} & 2 \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \end{pmatrix}_{r\theta\phi}$$

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

hydraulic diameter (general), $D_H \equiv \frac{4A_{xs}}{\varphi}$

$$Po = Re_{D_H} f_{D_H}$$

$$\frac{1}{\sqrt{f_{D_H}}} = 4.0 \log \left(\frac{Re_{D_H} \sqrt{f_{D_H}}}{\frac{Po_{duct}}{16}} \right) - 0.40$$

Geometry	Po
Circle	16
Equilateral triangle	13.33
Slit	24
Ellipse (a,b)	$\frac{32\pi^2}{\varphi} (a^2 + b^2)$

void fraction, $\varepsilon = \frac{\text{empty bed volume}}{\text{total bed volume}}$

$$a_v = \frac{\text{total partical surface area}}{\text{particle volume}}$$

$$\text{superficial velocity, } v_0 = \frac{Q}{V/L}$$

hydraulic diameter for packed bed, $D_H = \frac{4\varepsilon}{(1-\varepsilon)a_v}$

Reynolds number for packed bed = $\frac{\rho(v_0/\varepsilon)D_H}{\mu}$

friction factor for packed bed = $\left(\frac{\Delta p}{L}\right) \left(\frac{D_H \varepsilon^2}{2\rho v_0^2}\right)$