



CM3110
Transport I
Part I: Fluid Mechanics



Michigan Tech



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 Michigan Technological University

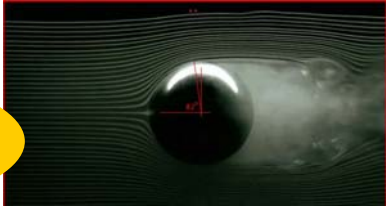


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Lecture 8

CM3110
Transport Processes and Unit Operations I
Fluid Mechanics
Microscopic Momentum Balances



Let's take stock

1. Control volumes
2. Coordinate systems
3. Continuity equation (microscopic mass balance)
4. Navier-Stokes (microscopic momentum balance)
5. Newton's law of viscosity
6. Boundary conditions
7. Solving differential equations
8. Calculate quantities of interest

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

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Can we apply this modeling method to more complex problems?

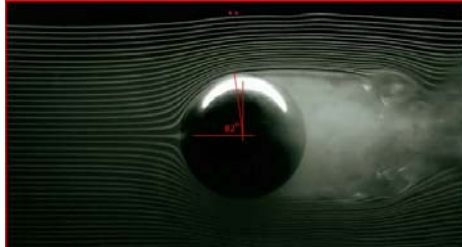
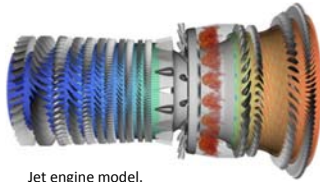


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Jet engine model.
Image from: www.stanford.edu

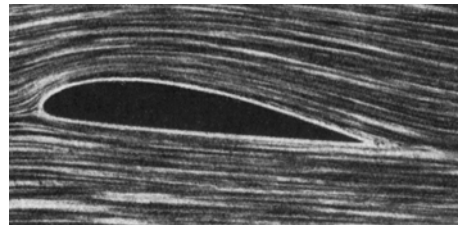


Image from: Prandtl and Tiejens (1929); p141 of our text

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(let's try)

To solve for complex flow fields:

1. Sketch and simplify if possible
2. Use convenient coordinate system
 - Match system symmetries
 - Minimize the number of components of velocity
3. Continuity equation (microscopic mass balance)
 - Use table for components
4. Navier-Stokes (microscopic momentum balance)
 - Use table for components
5. Newton's law of viscosity
 - ????
6. Boundary conditions
 - ????
7. Solve differential equations
 - Use advanced methods
 - Use computers
8. Calculate quantities of interest
 - ????

Continuity equation
(microscopic mass balance)

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

Navier-Stokes equation
(microscopic momentum balance)

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

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(let's try)

To solve for complex flow fields:

1. Sketch and simplify if possible
2. Use convenient coordinate system
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3. Continuity equation (microscopic mass balance)
 - Use table for components
4. Navier-Stokes (microscopic momentum balance)
 - Use table for components
5. Newton's law of viscosity
 - ???? *What should we use in the complex case?*
6. Boundary conditions
 - ???? *What should we use in the complex case?*
7. Solve differential equations
 - Use advanced methods
 - Use computers
8. Calculate quantities of interest
 - ???? *What should we use in the complex case?*

**Continuity equation
(microscopic mass balance)**

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

**Navier-Stokes equation
(microscopic momentum balance)**

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

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Complex flow fields

Three questions remain: *How do we handle the following:*

1. Newton's law of viscosity
 - ????
2. Boundary conditions
 - ????
3. Calculations of quantities of interest
 - ????

- Flow rate, Q
 - Average velocity, $\langle v \rangle$
 - Forces due to fluids
 - Torques due to fluids

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Complex flow fields

1. Newton's law of viscosity

$$\tilde{\tau}_{yz} = \mu \left(\frac{dv_z}{dy} \right) \quad \text{Newton's Law of Viscosity}$$

(Scalar relationship; one coordinate system)

- Works for unidirectional flow
-



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Complex flow fields

1. Newton's law of viscosity

$$\tilde{\tau}_{yz} = \mu \left(\frac{dv_z}{dy} \right) \quad \text{Newton's Law of Viscosity}$$

(Scalar relationship; one coordinate system)

- Works for unidirectional flow
(that's it)



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Complex flow fields

In flows other than unidirectional flow, we need the more general relationship: the *Newtonian Constitutive Equation*

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

Newtonian Constitutive Equation

(Tensor relationship; all coordinate systems)

$$\underline{\underline{\tilde{\tau}}} = \begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz}$$

In general, there are 9 components of stress at every location in a fluid; the $\underline{\underline{\tilde{\tau}}}$ sign convention is tension is positive

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Complex flow fields

Both expressions give the link between:

- **Deformation** (change of shape)
- and **Stress**

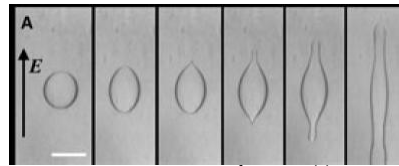


Image from: www.labspace.net

$$\tilde{\tau}_{yz} = \mu \left(\frac{dv_z}{dy} \right)$$

Newton's Law of Viscosity
(unidirectional flow)

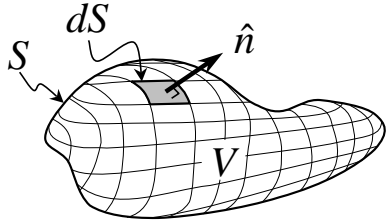
$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

Newtonian Constitutive Equation
(all types of flow fields)

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Equation of Motion



microscopic **momentum** balance written on an arbitrarily shaped volume, V , enclosed by a surface, S

We used $\underline{\underline{\tilde{\tau}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$ here:

Gibbs notation: $\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \nabla \cdot \underline{\underline{\tilde{\tau}}} + \rho \underline{g}$ **general fluid**

Gibbs notation: $\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$ **Newtonian fluid**

Navier-Stokes Equation

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Newtonian Constitutive Equation

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

Gives the link between deformation and stress.

$$\nabla \underline{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$$\nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

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$$\underline{\underline{\tilde{\tau}}} = \mu \dot{\underline{\underline{\gamma}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

Newtonian Constitutive Equation

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz}$$

$= \mu$

$$\begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz} \Rightarrow$

$\tilde{\tau}_{xz} = \mu \left(\frac{dv_z}{dx} \right)$

Newton's law of viscosity is a **special case** of the Newtonian Constitutive equation.

(Unidirectional flow)

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The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Total stress

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

Pressure (isotropic), $-p\underline{\underline{I}}$
 Viscous stress (anisotropic), $\underline{\underline{\tilde{\tau}}}$

For other coordinate systems, use the handout

$$\begin{pmatrix} \tilde{\Pi}_{xx} & \tilde{\Pi}_{xy} & \tilde{\Pi}_{xz} \\ \tilde{\Pi}_{yx} & \tilde{\Pi}_{yy} & \tilde{\Pi}_{yz} \\ \tilde{\Pi}_{zx} & \tilde{\Pi}_{zy} & \tilde{\Pi}_{zz} \end{pmatrix}_{xyz}$$

$=$

$$\begin{pmatrix} 2\mu \frac{\partial v_x}{\partial x} - p & \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & 2\mu \frac{\partial v_y}{\partial y} - p & \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \\ \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) & \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) & 2\mu \frac{\partial v_z}{\partial z} - p \end{pmatrix}_{xyz}$$

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The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Total stress

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \mu(\nabla\underline{v} + (\nabla\underline{v})^T)$$

For other coordinate systems, use the handout

Pressure only acts as a *normal* (perpendicular) push=negative pull

$$\begin{pmatrix} \tilde{\Pi}_{xx} & \tilde{\Pi}_{xy} & \tilde{\Pi}_{xz} \\ \tilde{\Pi}_{yx} & \tilde{\Pi}_{yy} & \tilde{\Pi}_{yz} \\ \tilde{\Pi}_{zx} & \tilde{\Pi}_{zy} & \tilde{\Pi}_{zz} \end{pmatrix}_{xyz} = \begin{pmatrix} 2\mu \frac{\partial v_x}{\partial x} - p & \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \\ \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) & 2\mu \frac{\partial v_y}{\partial y} - p & \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \\ \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) & \mu \left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) & 2\mu \frac{\partial v_z}{\partial z} - p \end{pmatrix}_{xyz}$$

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Newtonian Constitutive Equation

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu(\nabla\underline{v} + (\nabla\underline{v})^T)$$

Gives the link between:

- Deformation (change of shape)
- and Stress

Notes:

- The viscous stresses are due to molecular forces
- How deformation and stress are linked depends on the molecules
- Some molecules do not follow the Newtonian Constitutive Equation
- Rheology! (Non-Newtonian Fluid Mechanics)

(CM4650 Polymer Rheology; spring of even years)

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Boundary Conditions

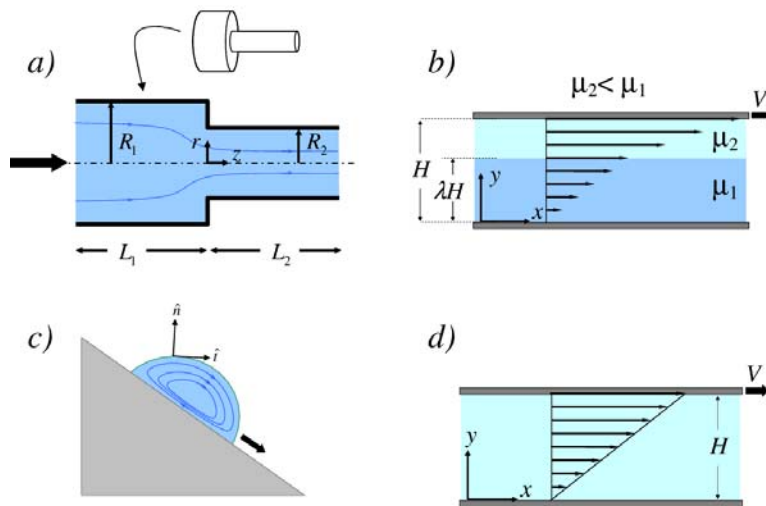


1. No slip
2. Symmetry
3. Extrema
4. Matching velocity, stress between fluids

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Boundary Conditions

(Ex 6.5, p464)



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How do we calculate quantities of interest?



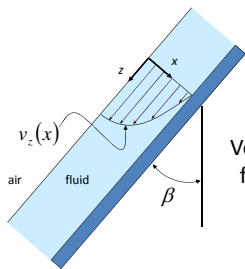
Image from: www.seriouswheels.com

1. Calculate **flow rate**
2. Calculate **average velocity**
3. Express **forces on surfaces** due to fluids
4. Express **torques on surfaces** due to fluids

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Engineering Quantities of Interest

Our strategy has been to develop the equation for each special case.



Average velocity

$$\langle v_z \rangle = \frac{\int_0^W \int_0^H v_z dx dy}{\int_0^W \int_0^H dx dy}$$

H is the height of the film; W is the width

Volumetric flow rate

$$Q = \int_0^W \int_0^H v_z dx dy = WH \langle v_z \rangle$$

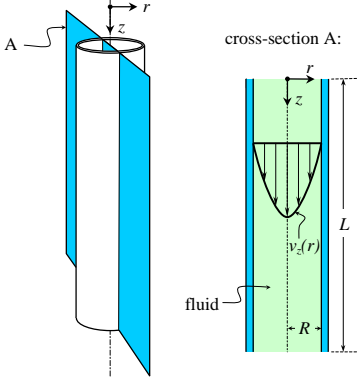
z -component of force on the wall

$$F_{z,on} = \int_0^L \int_0^W \tilde{\tau}_{xz} \Big|_{x=H} dy dz$$

(The expressions are different in different coordinate systems)

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Engineering Quantities of Interest
(tube flow)



Our strategy has been to develop the equation for each special case.

Average velocity $\langle v_z \rangle = \frac{\int_0^{2\pi} \int_0^R v_z r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta}$

Volumetric flow rate $Q = \int_0^{2\pi} \int_0^R v_z r dr d\theta = \pi R^2 \langle v_z \rangle$

z-component of force on the wall $F_{z,on} = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz} \Big|_{r=R} R d\theta dz$

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Engineering Quantities of Interest
(any flow)

In more complex flows, we can use general expressions that work in all cases.

average velocity $\langle v_z \rangle = \frac{\iint_S (\hat{n} \cdot \underline{v})|_{surface} dS}{\iint_S dS}$

volumetric flow rate $Q = \iint_S (\hat{n} \cdot \underline{v})|_{surface} dS$

Using the general formulas will help prevent errors.

Here, \hat{n} is the outwardly pointing unit normal of dS ; it points in the direction “through” S

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Engineering Quantities of Interest

(any flow)

volumetric flow rate

average velocity

z-component of force **on** the wall

In more complex flows, we can use general expressions that work in all cases.

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dS} = \frac{Q}{S}$$

$$F_{z,on} = \hat{e}_z \cdot \iint_S [\hat{n} \cdot (-p\underline{I} + \underline{\tilde{\tau}})]_{surface} dS$$

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Common surface shapes:

rectangular : $dS = dx dy$

circular top : $dS = r dr d\theta$

surface of cylinder : $dS = R d\theta dz$

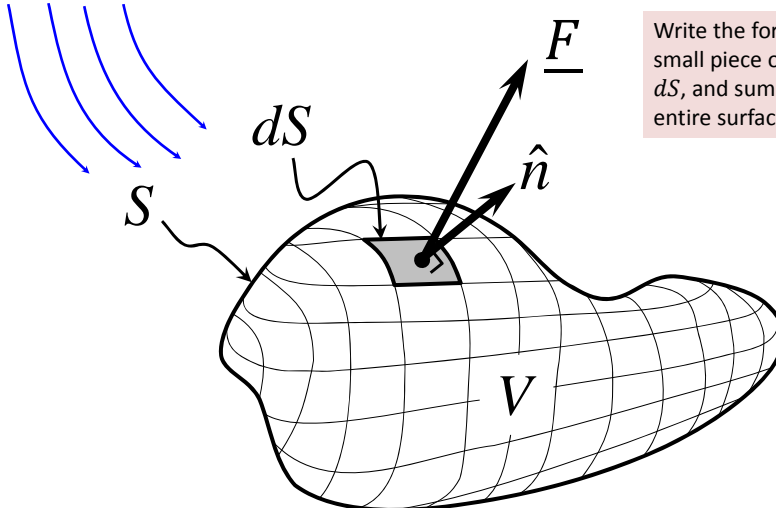
sphere : $dS = (R d\theta)(r \sin \theta d\phi) = R^2 \sin \theta d\theta d\phi$

Note: spherical coordinate system in use by fluid mechanics community uses $0 < \theta < \pi$ as the angle from the z-axis to the point.

(For more areas, see inside back cover of text)

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What is the general expression for fluid force on a surface?



Write the force **on** a small piece of surface dS , and sum over the entire surface.

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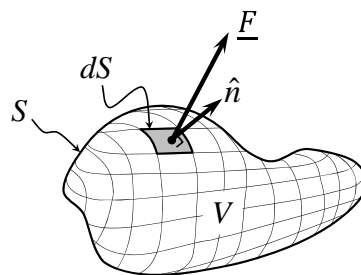
The stress tensor was invented to make this calculation easier.

Total stress tensor, $\underline{\underline{\Pi}}$:

$$\underline{\underline{\tilde{\Pi}}} \equiv -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$$

We can show:
(any flow, small surface)

Force on the surface $dS = \hat{n} \cdot \underline{\underline{\tilde{\Pi}}} dS$



This is the power of the stress tensor:
It allows us to calculate fluid forces on **any** surface.

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To get the total force, we integrate over the entire surface of interest.

Fluid force on the surface S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-p\underline{I} + \underline{\underline{\tilde{\tau}}} \right) \right]_{surface} dS$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz} = \iint_S (\hat{n}_x \hat{n}_y \hat{n}_z) \cdot \begin{pmatrix} -p + \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & -p + \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & -p + \tilde{\tau}_{zz} \end{pmatrix}_{xyz} dS$$

\hat{n} , $\underline{\underline{\tilde{\tau}}}$ and p evaluated at the surface dS

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Engineering Quantities of Interest

(any flow)

force **on** the surface, S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-p\underline{I} + \underline{\underline{\tilde{\tau}}} \right) \right]_{surface} dS$$

z-component of force on the surface, S

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{I} + \underline{\underline{\tilde{\tau}}} \right) \right]_{surface} dS$$

Using the general formulas will help prevent errors (like forgetting the pressure).

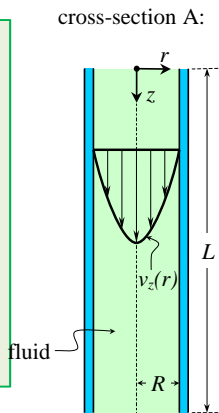
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Example 1: What is the force on the walls in Poiseuille flow of a Newtonian fluid:

$$p(z) = \frac{-(p_0 - p_L)}{L} z + p_0$$

$$\tilde{\tau}_{rz}(r) = \frac{-(\rho g L + p_0 - p_L)}{2L} r$$

$$v(r) = \frac{R^2(\rho g L + p_0 - p_L)}{4\mu L} \left(1 - \left(\frac{r}{R}\right)^2\right)$$



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Poiseuille flow of a Newtonian fluid:

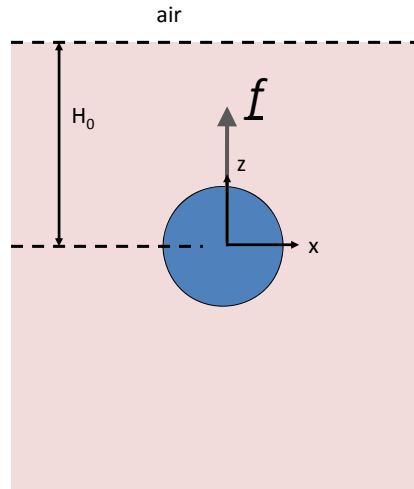
Force on the walls (general case):

$$F_z = ?$$

See hand notes

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Text:
 Example 4: In a liquid of density ρ , what is the net fluid force on a submerged sphere (a ball or a balloon)? What is the direction of the force and how does the magnitude of the fluid force vary with fluid density?



From expression for force due to fluid, obtain (spherical coordinates):

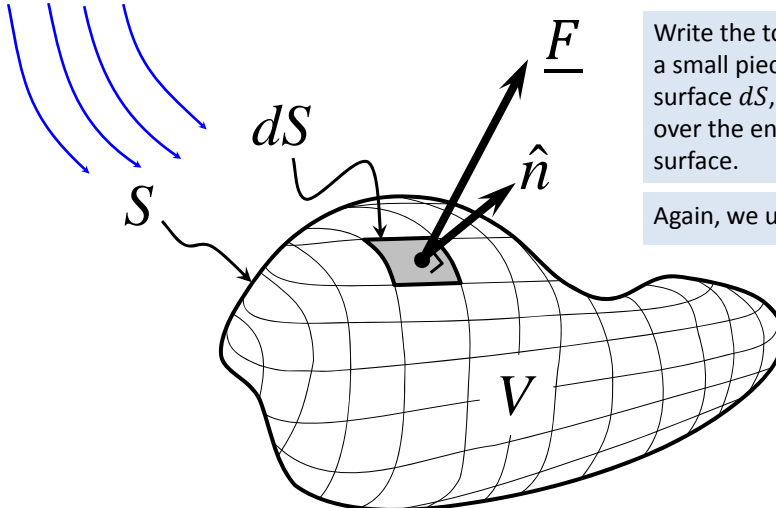
Total fluid force on a surface:

$$\underline{F} = \iint_S [\hat{n} \cdot \underline{\tilde{\Pi}}]_{surface} dS$$

$$\underline{F} = -\rho g R^2 \int_0^{2\pi} \int_0^{\pi} (H_0 - R \cos \theta) \hat{e}_r \sin \theta d\theta d\phi$$

Solution: (Ch4, p257).

What is the general expression for fluid torque on an object?



Write the torque on a small piece of surface dS , and sum over the entire surface.

Again, we use $\underline{\underline{\tilde{\Pi}}}$.

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What is the general expression for fluid torque on an object?

Total Fluid Torque on a macroscopic surface, S

$$\underline{T} = \iint_S [\underline{R} \times (\hat{n} \cdot \underline{\underline{\tilde{\Pi}}})]_{surface} dS$$

\underline{R} = lever arm (Points from axis of rotation to position where torque is applied, that is to dS)

$\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{I}$ = total stress tensor


\hat{n} = unit normal to dS in contact with the fluid

(All three may be a function of position)

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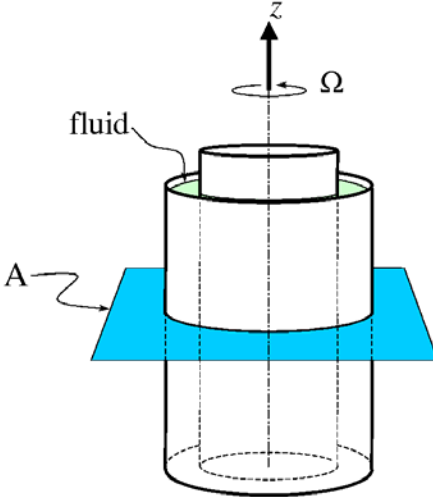
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Chapter 2 Redux Chapter 2 Redux Chapter 2

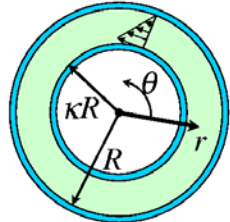


Text:

Example 5, Torque in Couette Flow: A cup-and-bob apparatus is widely used to measure viscosities for fluids. For the apparatus below, what is the **torque** needed to turn the inner cylinder (called the bob) at an angular speed of Ω ?




cross-section A:



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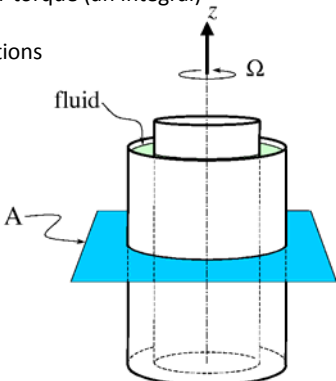
Chapter 2 Redux Chapter 2 Redux Chapter 2



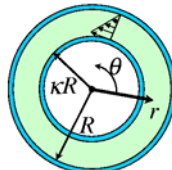
Torque in Couette Flow

Solution:

1. Solve for velocity field (microscopic momentum bal)
2. Calculate stress tensor
3. Formulate equation for torque (an integral)
4. Integrate
5. Apply boundary conditions



cross-section A:




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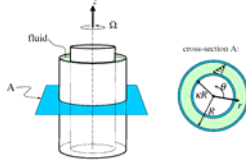
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Chapter 2 Redux Chapter 2 Redux Chapter 2

(Problem 6.22 p487)

Torque in Couette Flow Solution:





Velocity solution:

$$\underline{v} = \begin{pmatrix} 0 \\ \left(\frac{\kappa^2 \Omega R}{\kappa^2 - 1} \right) \left(\frac{r}{R} - \frac{R}{r} \right) \\ 0 \end{pmatrix}_{r\theta z}$$

$$\underline{\tilde{\tau}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T) \quad \underline{\tilde{\tau}} = \begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tilde{\tau}_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z}$$

$$\underline{\tilde{\Pi}} = \underline{\tilde{\tau}} - p\underline{I}$$

Total torque on a fluid surface:

$$\underline{T} = \iint_S [\underline{R} \times (\hat{n} \cdot \underline{\tilde{\Pi}})]_{surface} dS$$

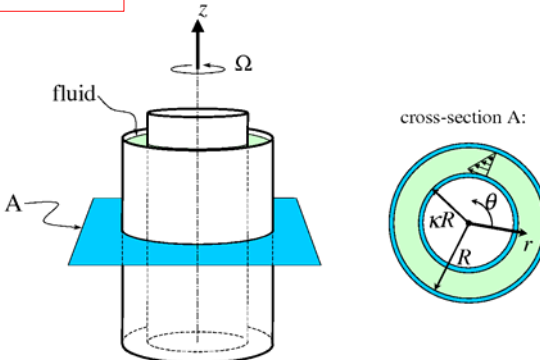
What is lever arm, \underline{R} ?

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Example: What is the torque on the inner cylinder in Couette flow?

$$\underline{T} = \iint_S [\underline{R} \times (\hat{n} \cdot \underline{\tilde{\Pi}})]_{surface} dS$$

See hand notes



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Engineering Quantities of Interest
(any flow)

volumetric flow rate through S

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity through S

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component of force on the surface S

$$F_z = \hat{e}_z \cdot \iint_S [\hat{n} \cdot (-p\underline{I} + \underline{\tilde{\tau}})]_{surface} dS$$

Total Fluid Torque on a surface, S

$$\underline{T} = \iint_S [\underline{R} \times (\hat{n} \cdot \underline{\tilde{\Pi}})]_{surface} dS$$

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Complex flow fields

Three questions remain: *How do we handle the following:*

1. Newton's law of viscosity
 - ????
2. Boundary conditions
 - ????
3. Calculations of quantities of interest
 - ????

- Flow rate, Q
- Average velocity, $\langle v \rangle$
- Forces due to fluids
- Torques due to fluids



Complex flow fields - SUMMARY

We handle these topics as follows:

1. *Newton's law of viscosity* → Use the Newtonian Constitutive Equation
2. *Boundary conditions* → Use vector relationships to write the boundary conditions for complex geometries
3. *Calculations of quantities of interest* → Use the general formulations (involve vector, matrix manipulations)

- Flow rate, Q
- Average velocity, $\langle v \rangle$
- Forces due to fluids
- Torques due to fluids