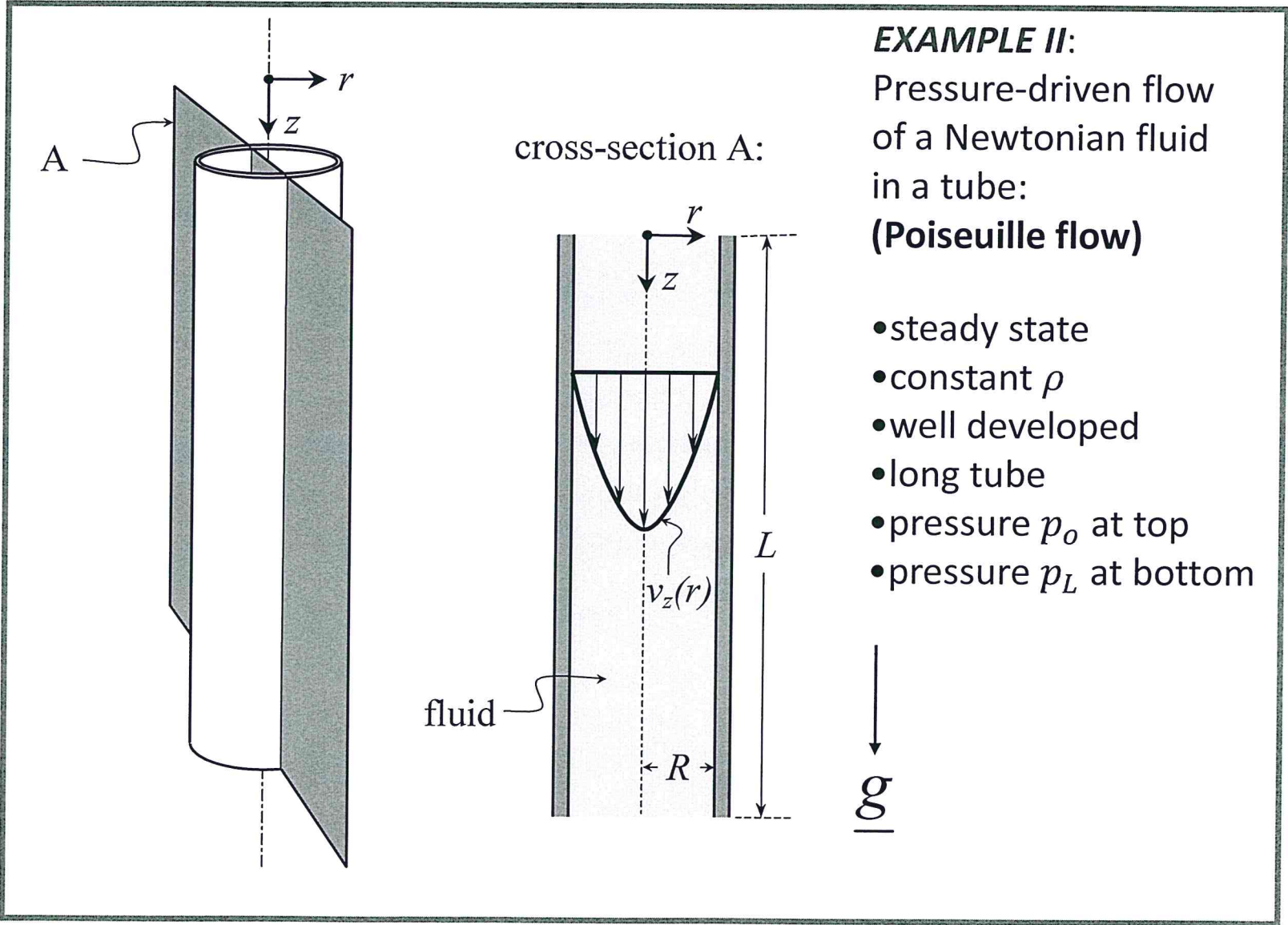


27 Sept 2017
FAM (A)

Calculate the steady state velocity field of an incompressible, Newtonian fluid flowing in a long pipe of circular cross section. The pressure at the entrance of the section of interest is P_0 and the pressure a distance L downstream is P_L .



$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}$$

Assume: unidirectional flow
in the z -direction

Step 1: MASS BALANCE

Continuity eqn

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \bar{\tau}_{xx}}{\partial x} + \frac{\partial \bar{\tau}_{yx}}{\partial y} + \frac{\partial \bar{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \bar{\tau}_{xy}}{\partial x} + \frac{\partial \bar{\tau}_{yy}}{\partial y} + \frac{\partial \bar{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \bar{\tau}_{xz}}{\partial x} + \frac{\partial \bar{\tau}_{yz}}{\partial y} + \frac{\partial \bar{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} - \frac{\bar{\tau}_{\theta\theta}}{r} + \frac{\partial \bar{\tau}_{zr}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) = \frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \bar{\tau}_{z\theta}}{\partial z} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r} \frac{\partial(r \bar{\tau}_{rz})}{\partial r} + \frac{1}{r} \frac{\partial \bar{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \bar{\tau}_{z z}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ &= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2} \frac{\partial(r^2 \bar{\tau}_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\theta r}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi r}}{\partial \phi} - \frac{\bar{\tau}_{\theta\theta} + \bar{\tau}_{\phi\phi}}{r} \right) + \rho g_r \end{aligned}$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} + \frac{v_\phi^2 \cot \theta}{r} \right)$$

$$= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\theta}}{\partial \phi} + \frac{\bar{\tau}_{\theta r} - \bar{\tau}_{r\theta}}{r} - \frac{\bar{\tau}_{\phi\phi} \cot \theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot \theta}{r} \right)$$

$$= -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3} \frac{\partial(r^3 \bar{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \bar{\tau}_{\phi\phi}}{\partial \phi} + \frac{\bar{\tau}_{\phi r} - \bar{\tau}_{r\phi}}{r} + \frac{\bar{\tau}_{\theta\theta} \cot \theta}{r} \right) + \rho g_\phi$$

(D)

Step 2: Micro Momentum Balance (Navier-Stokes)

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_r v_z}{r} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r} \left(\frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\sin \theta} \frac{\partial \theta}{\partial \phi} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r \sin \theta} + \frac{v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial (v_\phi \sin \theta)}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi$$

Note: the r -component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r} \nabla \cdot \underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.
2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

$$\vec{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$$

①

N-S Egn

r-component:

$$0 = -\frac{\partial p}{\partial r}$$

\Rightarrow P is not a function of r

θ -component

$$0 = -\frac{1}{r} \frac{\partial p}{\partial \theta}$$

\Rightarrow P is not a function of θ

②

NS z -component

$$0 = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) \right)$$

$$+ \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \rho g$$

$P(z)$ only

θ -symmetry

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z(r) \end{pmatrix}_{r\theta z}$$

3

If we can separate the two variables:

$$\underbrace{f(r)}_{\text{only function of } r} = \underbrace{W(z)}_{\text{only a function of } z} = \lambda$$

must be equal to some constant

$$f(r) = \lambda$$

$$W(z) = \lambda$$

Solve separately.

$$\underbrace{\frac{dP}{dz} - \rho g}_{\text{only a function of } z} = \underbrace{\mu \left(\frac{L}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) \right)}_{\text{only a function of } r} \quad \textcircled{4}$$

only a function of z
only a function of r

Solve $P(z)$:

$$\frac{dP}{dz} = (\lambda + \rho g)$$

$$P = (\lambda + \rho g)z + C$$

BC: $z = 0 \quad P = P_0$
 $z = L \quad P = P_L$

2 EQN
2 unknowns
SOLVE

$$P_0 = \cancel{(\lambda + \rho g)(z)} + C_1 \Rightarrow$$
$$P_L = (\lambda + \rho g)(L) + C_1$$

$$\boxed{C_1 = P_0}$$

$$P_L - P_0 = (\lambda + \rho g)L$$

~~*~~
$$\left(\frac{P_L - P_0}{L} \right) - \rho g = \lambda$$

plus
back;
solve

$$P = \left(\frac{P_L - P_0}{L} \right) z + P_0$$

$$\frac{\lambda}{\mu} = \frac{1}{r} \frac{d}{dr} \left(r \left(\frac{d\psi}{dr} \right) \right) \quad \text{--- } \psi \quad \textcircled{6}$$

$$\left(\frac{\lambda}{\mu} \right) r = \frac{d}{dr} (\psi)$$

$$\frac{d\psi}{dr} = \left(\frac{\lambda}{\mu} \right) r$$

$$\psi = \left(\frac{\lambda}{\mu} \right) \frac{r^2}{2} + C_2$$

$$r \frac{d\psi}{dr} = \left(\frac{\lambda}{\mu} \right) r^2 + C_2$$

$$\frac{dV_z}{dr} = \left(\frac{\lambda}{4\mu}\right) r + \frac{C_2}{r}$$

⑦

$$V_z = \left(\frac{\lambda}{4\mu}\right) \frac{r^2}{2} + C_2 (\ln r) + C_3$$

B.C. $r = R$

$$V_z = 0$$

no slip
boundary
condition

$$\frac{dV_z}{dr} = 0$$

$$\Rightarrow C_2 = 0$$

fluid speed
is max at
center

$r = 0$

$$\Rightarrow C_2 = 0$$

$$0 = \frac{\lambda}{4\mu} R^2 + C_3$$

$$\Rightarrow C_3 = -\frac{\lambda}{4\mu} R^2$$

put
back in.
Done.



The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

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Cartesian Coordinates

$$\underline{\tau} = \mu \begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2 \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2 \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tilde{\tau}_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{r} \frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial r} \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial r} & 2 \frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tilde{\tau}_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \mu \begin{pmatrix} 2 \frac{\partial v_r}{\partial r} & r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \\ r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} & 2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) & \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} & 2 \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) \end{pmatrix}_{r\theta\phi}$$

0 0 0 / $\mu \frac{\partial v}{\partial r}$ / $r \frac{\partial v_\theta}{\partial r}$ / $r \frac{\partial v_\phi}{\partial r}$

continuity

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)