CM3110
Transport I
Part I: Fluid Mechanics

Complex Flows

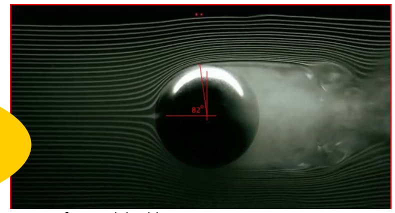
Image from: gkdot.blogspot.com


Image from: www.g4tv.com

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Lecture 8

CM3110
Transport Processes and Unit Operations I

Fluid Mechanics
Microscopic Momentum Balances

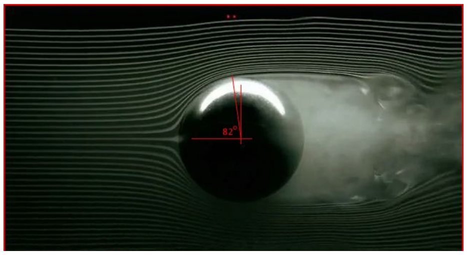
Let's take stock

1. Control volumes
2. Coordinate systems
3. Continuity equation (microscopic mass balance)
4. Navier-Stokes (microscopic momentum balance)
5. Newton's law of viscosity
6. Boundary conditions
7. Solving differential equations
8. Calculate quantities of interest

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Can we apply this modeling method to more complex problems?



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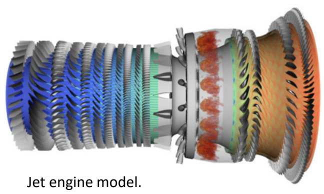

Jet engine model.
Image from: www.stanford.edu

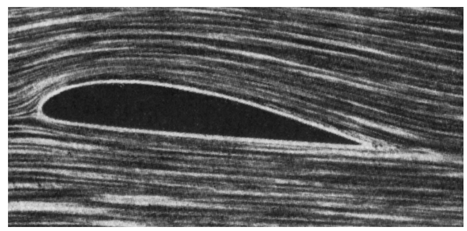

Image from: Prandtl and Tietjens (1929); p141 of our text

 3

(let's try)

To solve for complex flow fields:

1. Sketch and simplify if possible
2. Use convenient coordinate system
   - Match system symmetries
   - Minimize the number of components of velocity
3. Continuity equation (microscopic mass balance)
   - Use table for components
4. Navier-Stokes (microscopic momentum balance)
   - Use table for components
5. Newton's law of viscosity
   - ????
6. Boundary conditions
   - ????
7. Solve differential equations
   - Use advanced methods
   - Use computers
8. Calculate quantities of interest
   - ????

Continuity equation (microscopic mass balance)

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

Navier-Stokes equation (microscopic momentum balance)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

 4

(let's try)

To solve for complex flow fields:

1. Sketch and simplify if possible
2. Use convenient coordinate system
   - Match system symmetries
   - Minimize the number of components of velocity
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   - Use table for components
4. Navier-Stokes (microscopic momentum balance)
   - Use table for components
5. Newton's law of viscosity
   - ???? What should we use in the complex case?
6. Boundary conditions
   - ???? What should we use in the complex case?
7. Solve differential equations
   - Use advanced methods
   - Use computers
8. Calculate quantities of interest
   - ???? What should we use in the complex case?

Continuity equation (microscopic mass balance)

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$$

Navier-Stokes equation (microscopic momentum balance)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = \nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

5

Complex flow fields

Three questions remain: *How do we handle the following:*

1. Newton's law of viscosity
   - ????
2. Boundary conditions
   - ????
3. Calculations of quantities of interest
   - ????

- Flow rate, Q
- Average velocity, $\langle v \rangle$
- Forces due to fluids
- Torques due to fluids

6

Complex flow fields

1. Newton's law of viscosity

$$\tilde{\tau}_{yz} = \mu \left(\frac{dv_z}{dy} \right)$$

Newton's Law of Viscosity

(Scalar relationship; one coordinate system)

- Works for unidirectional flow
-

7

Complex flow fields

1. Newton's law of viscosity

$$\tilde{\tau}_{yz} = \mu \left(\frac{dv_z}{dy} \right)$$

Newton's Law of Viscosity

(Scalar relationship; one coordinate system)

- Works for unidirectional flow

(that's it)

8

Complex flow fields

In flows other than unidirectional flow, we need the more general relationship: the *Newtonian Constitutive Equation*

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Newtonian Constitutive Equation

(Tensor relationship; all coordinate systems)

$$\underline{\underline{\tilde{\tau}}} = \begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz}$$

In general, there are 9 components of stress at every location in a fluid; the $\underline{\underline{\tilde{\tau}}}$ sign convention is tension is positive

9

Complex flow fields

Both expressions give the link between:

- ***Deformation*** (change of shape)
- and ***Stress***

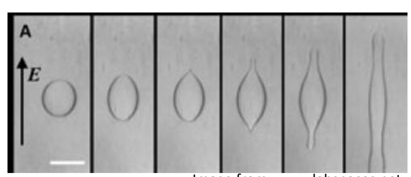


Image from: www.labspaces.net

$$\tilde{\tau}_{yz} = \mu\left(\frac{dv_z}{dy}\right)$$

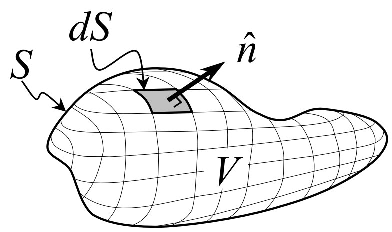
Newton's Law of Viscosity
(unidirectional flow)

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Newtonian Constitutive Equation
(all types of flow fields)

10

Equation of Motion



microscopic **momentum** balance written on an arbitrarily shaped volume, *V*, enclosed by a surface, *S*

We used $\underline{\underline{\tilde{\tau}}} = \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$ here:

Gibbs notation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \nabla\cdot\underline{\underline{\tilde{\tau}}} + \rho\underline{g}$ **general fluid**

Gibbs notation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$ **Newtonian fluid**

Navier-Stokes Equation

11

Newtonian Constitutive Equation

$$\underline{\underline{\tilde{\tau}}} = \mu\underline{\underline{\dot{\gamma}}} = \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$$

Gives the link between deformation and stress.

$$\nabla\underline{v} = \begin{pmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_y}{\partial x} & \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} & \frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z} & \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$$\nabla\underline{v} + \left(\nabla\underline{v}\right)^T = \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

12

$$\underline{\underline{\tilde{\tau}}} = \mu\dot{\underline{\underline{\gamma}}} = \mu(\nabla\underline{v} + (\nabla\underline{v})^T)$$

Newtonian Constitutive Equation

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz} \Rightarrow \quad \tilde{\tau}_{xz} = \mu\left(\frac{dv_z}{dx}\right)$$

*Newton's law of viscosity is a **special case** of the Newtonian Constitutive equation.*

(Unidirectional flow)

13

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

For other coordinate systems, use the handout

Total stress

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$$

Pressure (isotropic), $-p\underline{\underline{I}}$

Viscous stress (anisotropic), $\underline{\underline{\tilde{\tau}}}$

$$\begin{pmatrix} \tilde{\Pi}_{xx} & \tilde{\Pi}_{xy} & \tilde{\Pi}_{xz} \\ \tilde{\Pi}_{yx} & \tilde{\Pi}_{yy} & \tilde{\Pi}_{yz} \\ \tilde{\Pi}_{zx} & \tilde{\Pi}_{zy} & \tilde{\Pi}_{zz} \end{pmatrix}_{xyz} = \begin{pmatrix} 2\mu\frac{\partial v_x}{\partial x} - p & \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & 2\mu\frac{\partial v_y}{\partial y} - p & \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & 2\mu\frac{\partial v_z}{\partial z} - p \end{pmatrix}_{xyz}$$

14

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

For other coordinate systems, use the handout

Total stress

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Pressure only acts as a *normal* (perpendicular) push=negative pull

$$\begin{pmatrix} \tilde{\Pi}_{xx} & \tilde{\Pi}_{xy} & \tilde{\Pi}_{xz} \\ \tilde{\Pi}_{yx} & \tilde{\Pi}_{yy} & \tilde{\Pi}_{yz} \\ \tilde{\Pi}_{zx} & \tilde{\Pi}_{zy} & \tilde{\Pi}_{zz} \end{pmatrix}_{xyz} = \begin{pmatrix} 2\mu\frac{\partial v_x}{\partial x} - p & \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & 2\mu\frac{\partial v_y}{\partial y} - p & \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) \\ \mu\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) & \mu\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & 2\mu\frac{\partial v_z}{\partial z} - p \end{pmatrix}_{xyz}$$

15

Newtonian Constitutive Equation

$$\underline{\underline{\tilde{\tau}}} = \mu\underline{\underline{\dot{\gamma}}} = \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Gives the link between:

- *Deformation* (change of shape)
- and *Stress*

Notes:

- The viscous stresses are due to molecular forces
- How deformation and stress are linked depends on the chemistry
- Some materials do not follow the Newtonian Constitutive Equation
- Rheology! (Non-Newtonian Fluid Mechanics)

(CM4650 Polymer Rheology; spring of even years)

16

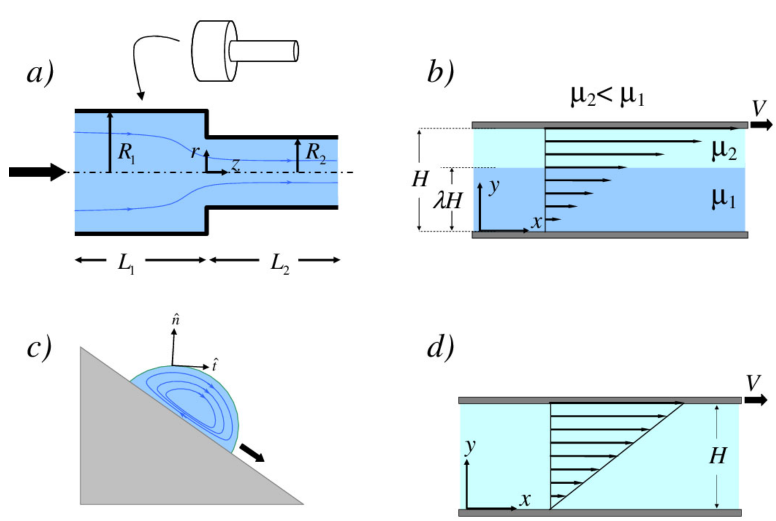
Boundary Conditions

1. No slip
2. Symmetry
3. Extrema
4. Matching velocity, stress between fluids

17

Boundary Conditions

(Ex 6.5, p464)



18

How do we calculate quantities of interest?

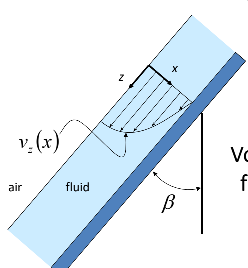
Image from: www.seriouswheels.com

1. Calculate **flow rate**
2. Calculate **average velocity**
3. Express **forces on surfaces** due to fluids
4. Express **torques on surfaces** due to fluids

 19

Engineering Quantities of Interest

Our strategy has been to develop the equation for each special case.



Average velocity

$$\langle v_z\rangle = \frac{\int_0^W \int_0^H v_z dxdy}{\int_0^W \int_0^H dxdy}$$

H is the height of the film; W is the width

Volumetric flow rate

$$Q = \int_0^W \int_0^H v_z\, dxdy = WH\langle v_z\rangle$$

z-component of force **on** the wall

$$F_{z,on} = \int_0^L \int_0^W \tilde{\tau}_{xz}\Big|_{x=H} dydz$$

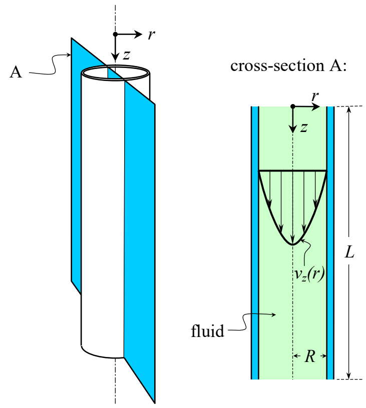
(The expressions are different in different coordinate systems)

 20

Engineering Quantities of Interest

(tube flow)

Our strategy has been to develop the equation for each special case.



Average velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta}$$

Volumetric flow rate

$$Q\rangle = \int_0^{2\pi}\int_0^{R} v_z r dr d\theta = \pi R^2 \langle v_z \rangle$$

z-component of force **on** the wall

$$F_{z,on} = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz}\Big|_{r=R} R d\theta dz$$

Engineering Quantities of Interest

(any flow)

In more complex flows, we can use general expressions that work in all cases.

average velocity

$$\langle v_z \rangle = \frac{\iint_S (\hat{n} \cdot \underline{v})|_{surface} dS}{\iint_S dS}$$

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v})\Big|_{surface} dS$$

Using the general formulas will help prevent errors.

Here, $\hat{n}$ is the outwardly pointing unit normal of dS; it points in the direction "through" S

Engineering Quantities of Interest

(any flow)

In more complex flows, we can use general expressions that work in all cases.

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dS} = \frac{Q}{S}$$

z-component of force **on** the wall

$$F_{z,on} = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)\right]_{surface} dS$$

23

Michigan Tech

Common surface shapes:

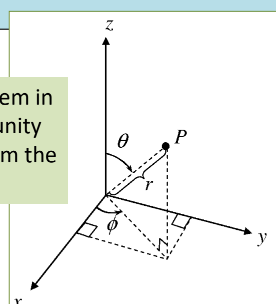
rectangular : $dS = dxdy$

circular top : $dS = r\,drd\theta$

surface of cylinder : $dS = Rd\theta dz$

sphere : $dS = (Rd\theta)(r\sin\theta\, d\phi) = R^2 \sin\theta\, d\theta d\phi$

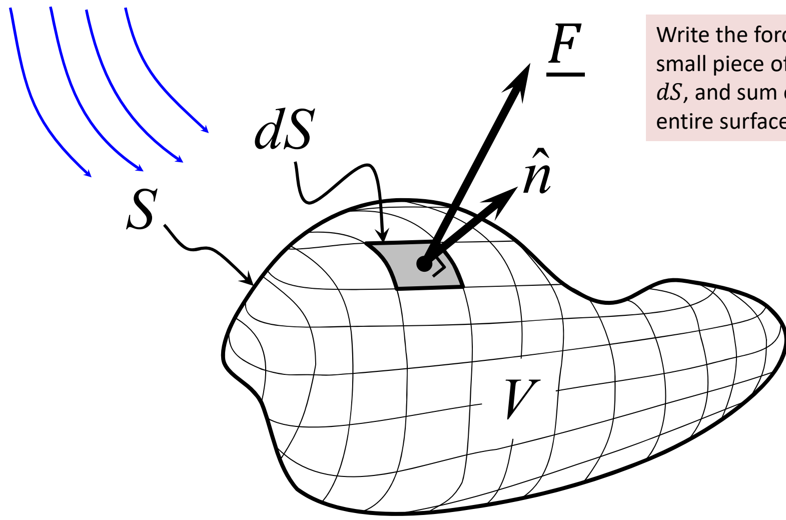
Note: spherical coordinate system in use by fluid mechanics community uses $0 < \theta < \pi$ as the angle from the z=axis to the point.



(Fore more areas, see inside back cover of text)

24

What is the general expression for fluid force on a surface?



Write the force **on** a small piece of surface dS, and sum over the entire surface.

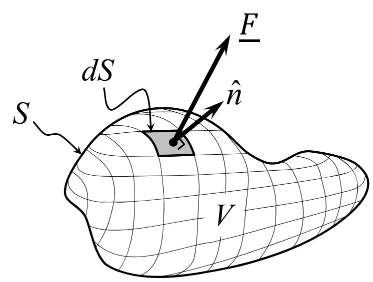
The stress tensor was invented to make this calculation easier.

Total stress tensor, $\underline{\underline{\tilde{\Pi}}}$:

$$\underline{\underline{\tilde{\Pi}}} \equiv -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$$

We can show:
(any flow, small surface)

Force on the surface dS $= \hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\, dS$



This is the power of the stress tensor: It allows us to calculate fluid forces on ***any*** surface.

To get the total force, we integrate over the entire surface of interest.

Fluid force on the surface S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \overbrace{\left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)}^{\underline{\underline{\tilde{\Pi}}}}\right]_{surface} dS$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz} = \underbrace{\iint_S (\hat{n}_x \quad \hat{n}_y \quad \hat{n}_z) \cdot \begin{pmatrix} -p + \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & -p + \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & -p + \tilde{\tau}_{zz} \end{pmatrix}_{xyz}}_{} dS$$

$\hat{n}$, $\underline{\underline{\tilde{\tau}}}$ and p evaluated at the surface dS

 27

Engineering Quantities of Interest

(any flow)

force **on** the surface, S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)\right]_{surface} dS$$

z-component of force on the surface, S

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)\right]_{surface} dS$$

Using the general formulas will help prevent errors (like forgetting the pressure).

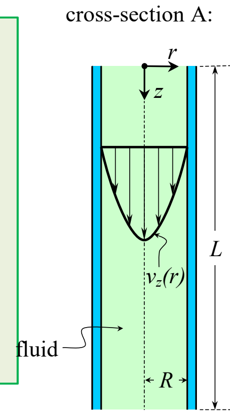
 28

Example 1: What is the force on the walls in Poiseuille flow of a Newtonian fluid (tube):

$$p(z) = \frac{-(p_0 - p_L)}{L} z + p_o$$

$$\tilde{\tau}_{rz}(r) = \frac{-(\rho g L + p_o - p_L)}{2L} r$$

$$v(r) = \frac{R^2(\rho g L + p_o - p_L)}{4\mu L}\left(1 - \left(\frac{r}{R}\right)^2\right)$$



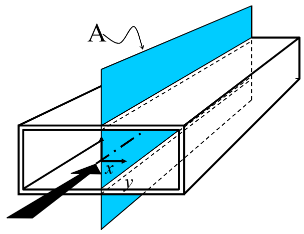
Poiseuille flow of a Newtonian fluid:

Force on the walls (general case):

$$F_z = ?$$

See hand notes

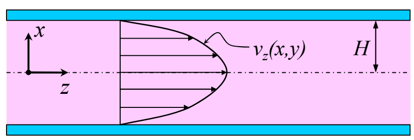
What about more complicated Newtonian problems?



EXAMPLE : Pressure-driven flow of a Newtonian fluid in a rectangular duct: **Poiseuille flow**

- steady state
- well developed
- long tube
- $P(0)=P_o$, $P(L)=P_L$

cross-section A:



$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z(x,y) \end{pmatrix}_{xyz}$$

Velocity varies in two directions

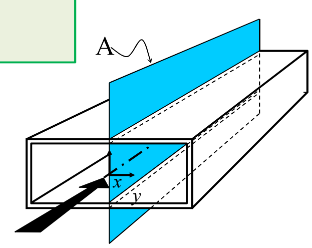
31

Example 2: What is the force on the walls in Poiseuille flow of a Newtonian fluid (rectangular):

$$p(z) = ?$$

$$\underline{\underline{\tau}}(x,y,z) = ?$$

$$\underline{v} = ?$$



Example 7.11, p549

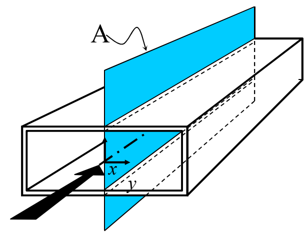
32

Poiseuille flow of a Newtonian fluid (rectangular duct):

Force on the walls (general case):

$$F_z = ?$$

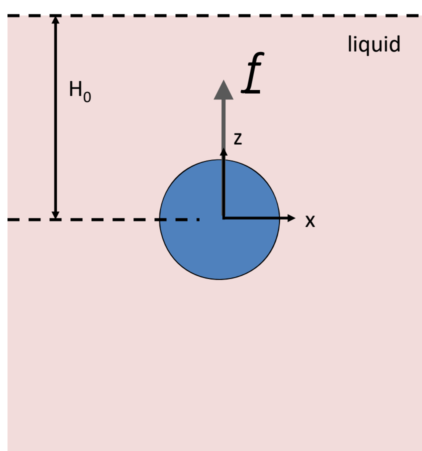
See hand notes



33

(p81) **Example 3:** (Ch 1, Ch 4, stretch) Michigan Tech

In a liquid of density ρ, what is the net fluid force on a submerged sphere (a ball or a balloon)? What is the direction of the force and how does the magnitude of the fluid force vary with fluid density?



34

Total fluid force on a surface:

$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

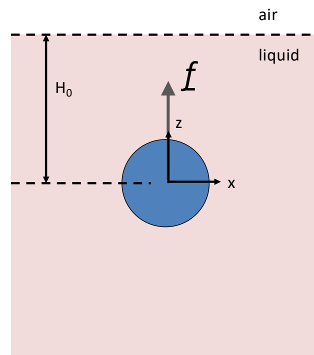
$\hat{n} = ?$

$\underline{\underline{\tilde{\Pi}}} = ?$

$dS = ?$

Location of $surface$?

Limits of integration?



See hand notes

35

From expression for force due to fluid, obtain (spherical coordinates):

Total fluid force on a surface:

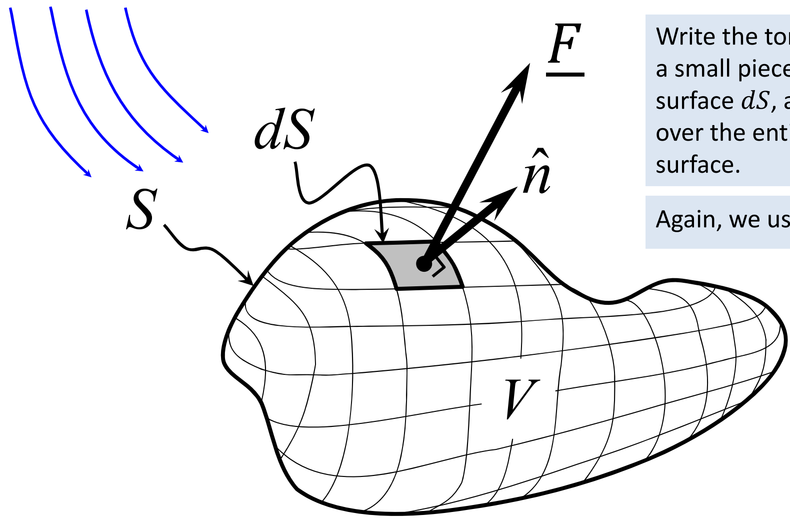
$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

$$\underline{F} = -\rho g R^2 \int_0^{2\pi} \int_0^{\pi} (H_0 - R\cos\theta)\hat{e}_r \sin\theta \, d\theta \, d\phi$$

Solution: (Ch4, p257).

36

What is the general expression for fluid torque on an object?



Write the torque on a small piece of surface dS, and sum over the entire surface.

Again, we use $\underline{\underline{\tilde{\Pi}}}$.

What is the general expression for fluid torque on an object?

Total Fluid Torque on a macroscopic surface, S

$$\underline{T} = \iint_S \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{surface} dS$$

$\underline{R}$ = lever arm (Points from axis of rotation to position where torque is applied, that is to dS)

$\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}}$ = total stress tensor

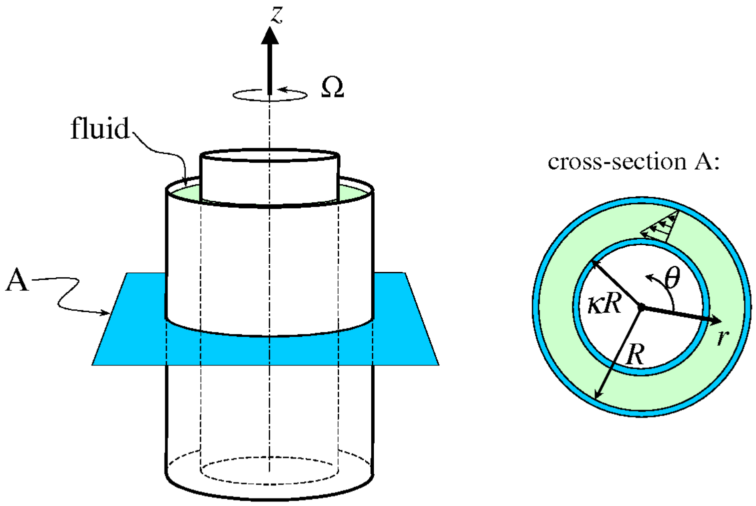
$\hat{n}$ = unit normal to dS in contact with the fluid

(All three may be a function of position)

38

Example 4: (Ch 6, stretch)

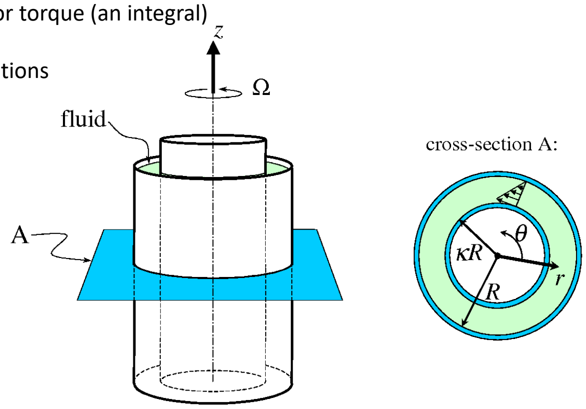
Torque in Couette Flow: A cup-and-bob apparatus is widely used to measure viscosities for fluids. For the apparatus below, what is the **torque** needed to turn the inner cylinder (called the bob) at an angular speed of Ω?



39

Torque in Couette Flow
Solution:

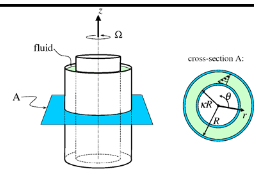
1. Solve for velocity field (microscopic momentum bal)
2. Calculate stress tensor
3. Formulate equation for torque (an integral)
4. Integrate
5. Apply boundary conditions



40

(Problem 6.22 p487

Torque in Couette Flow Solution:



Velocity solution:

$$\underline{v} = \begin{pmatrix} 0 \\ \left(\frac{\kappa^2 \Omega R}{\kappa^2 - 1}\right)\left(\frac{r}{R} - \frac{R}{r}\right) \\ 0 \end{pmatrix}_{r\theta z}$$

$$\tilde{\underline{\underline{\tau}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$$

$$\tilde{\underline{\underline{\Pi}}} = \tilde{\underline{\underline{\tau}}} - p\underline{\underline{I}}$$

$$\tilde{\underline{\underline{\tau}}} = \begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tilde{\tau}_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z}$$

Total torque on a fluid surface:

$$\underline{T} = \iint_S \left[\underline{R} \times \left(\hat{n} \cdot \tilde{\underline{\underline{\Pi}}}\right)\right]_{surface} dS$$

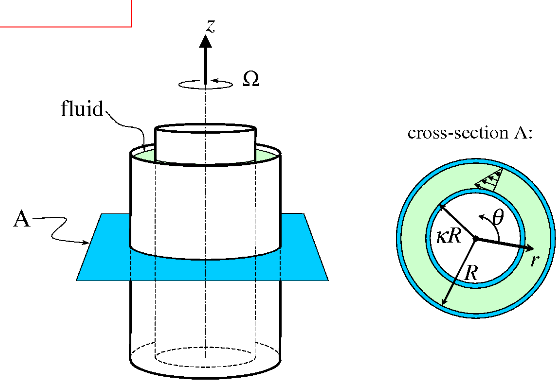
What is lever arm, $\underline{R}$?

41

Example 4: What is the torque on the inner cylinder in Couette flow?

$$\underline{T} = \iint_S \left[\underline{R} \times \left(\hat{n} \cdot \tilde{\underline{\underline{\Pi}}}\right)\right]_{surface} dS$$

Try it!



42

Engineering Quantities of Interest

(any flow)

volumetric flow rate through S

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity through S

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component of force on the surface S

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot (-p\underline{\underline{I}} + \tilde{\underline{\underline{\tau}}})\right]_{surface} dS$$

Total Fluid Torque on a surface, S

$$\underline{T} = \iint_S \left[\underline{R} \times (\hat{n} \cdot \tilde{\underline{\underline{\Pi}}})\right]_{surface} dS$$

43

Complex flow fields

Three questions remain: *How do we handle the following:*

1. Newton's law of viscosity
   - ????
2. Boundary conditions
   - ????
3. Calculations of quantities of interest
   - ????

- Flow rate, Q
- Average velocity, $\langle v \rangle$
- Forces due to fluids
- Torques due to fluids

44

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Complex flow fields - SUMMARY

We handle these topics as follows:

1. *Newton's law of viscosity* → Use the Newtonian Constitutive Equation
2. *Boundary conditions* → Use vector relationships to write the boundary conditions for complex geometries
3. *Calculations of quantities of interest* → Use the general formulations (involve vector, matrix manipulations)

- Flow rate, Q
- Average velocity, $\langle v \rangle$
- Forces due to fluids
- Torques due to fluids

45