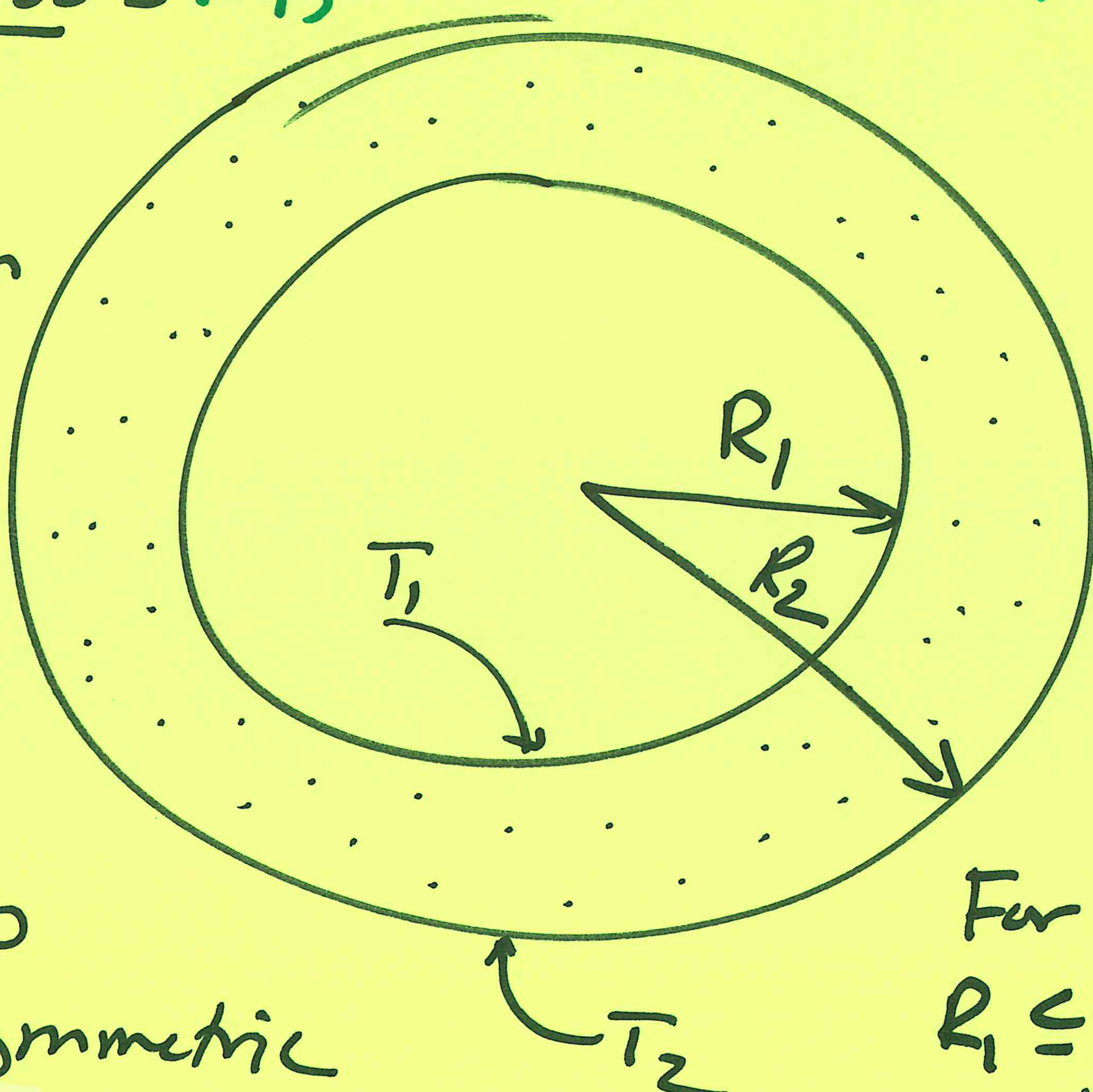
EXAMPLE 3 (+4) LEC 18 10-31-19 1

- radial heat conduction
- steady
- long



steady
long
Θ-symmetric

$T_1 > T_2$

For

$R_1 \le r \le R_2$

k = thermal conductivity

1D radial heat conduction (using temp version of E-Bal) 2

The **Equation of Energy** for systems with **constant k**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_r \frac{\partial T}{\partial r}} + \cancel{\frac{v_\theta}{r} \frac{\partial T}{\partial \theta}} + \cancel{v_z \frac{\partial T}{\partial z}} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \cancel{\frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}} \right) + \cancel{S_e}$$

SS

$v = 0$

symmetry

long

no rxn
no elec current

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right)$$
$$= k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S_e$$

$$0 = \cancel{k} \cancel{\left(\frac{1}{r}\right)} \frac{d}{dr} \Big(\underbrace{r \frac{dT}{dr}}_{\Phi \equiv} \Big)$$

$$\frac{d\Phi}{dr} = 0$$

$$\Phi = c_1 = r \frac{dT}{dr}$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

$$\frac{c_1}{r} = \frac{dT}{dr}$$

temp profile

$$\boxed{T = c_1 \ln r + c_2}$$

★ Temp profile is ln r

FLUX?

Furier's Law:

$$\frac{q_r}{A} = -k\frac{dT}{dr} = \boxed{-\frac{k c_1}{r}}$$

★ Flux goes like 1/r

BC: $r = R_1 \quad T = T_1$

$r = R_2 \quad T = T_2$

- steady
- 1D
- radial
- long, θ sym

substitute BCs:

$$T_1 = c_1 \ln R_1 + c_2$$
$$T_2 = c_1 \ln R_2 + c_2$$

2 eqns, 2 unknowns, solve for c_1, c_2

SOLUTION STEPS

subtract: $(T_1 - T_2) = c_1 \ln\left(\frac{R_1}{R_2}\right)$

$$\boxed{c_1 = \frac{T_1 - T_2}{\ln \frac{R_1}{R_2}}}$$

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Substitute into second eqn:

$$T_2 = C_1 \ln R_2 + C_2$$

$$T_2 = \frac{(T_1 - T_2)}{\ln \frac{R_1}{R_2}} \ln R_2 + C_2$$

$$\boxed{C_2 = T_2 - (T_1 - T_2)\frac{\ln R_2}{\ln R_1/R_2}}$$

substitute back into temp profile:

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$$T = c_1(\ln r) + c_2$$

$$T = \left(\frac{(T_1 - T_2)}{\ln R_1/R_2}\right)\ln r + T_2 - (T_1 - T_2)\left(\frac{\ln R_2}{\ln R_1/R_2}\right)$$

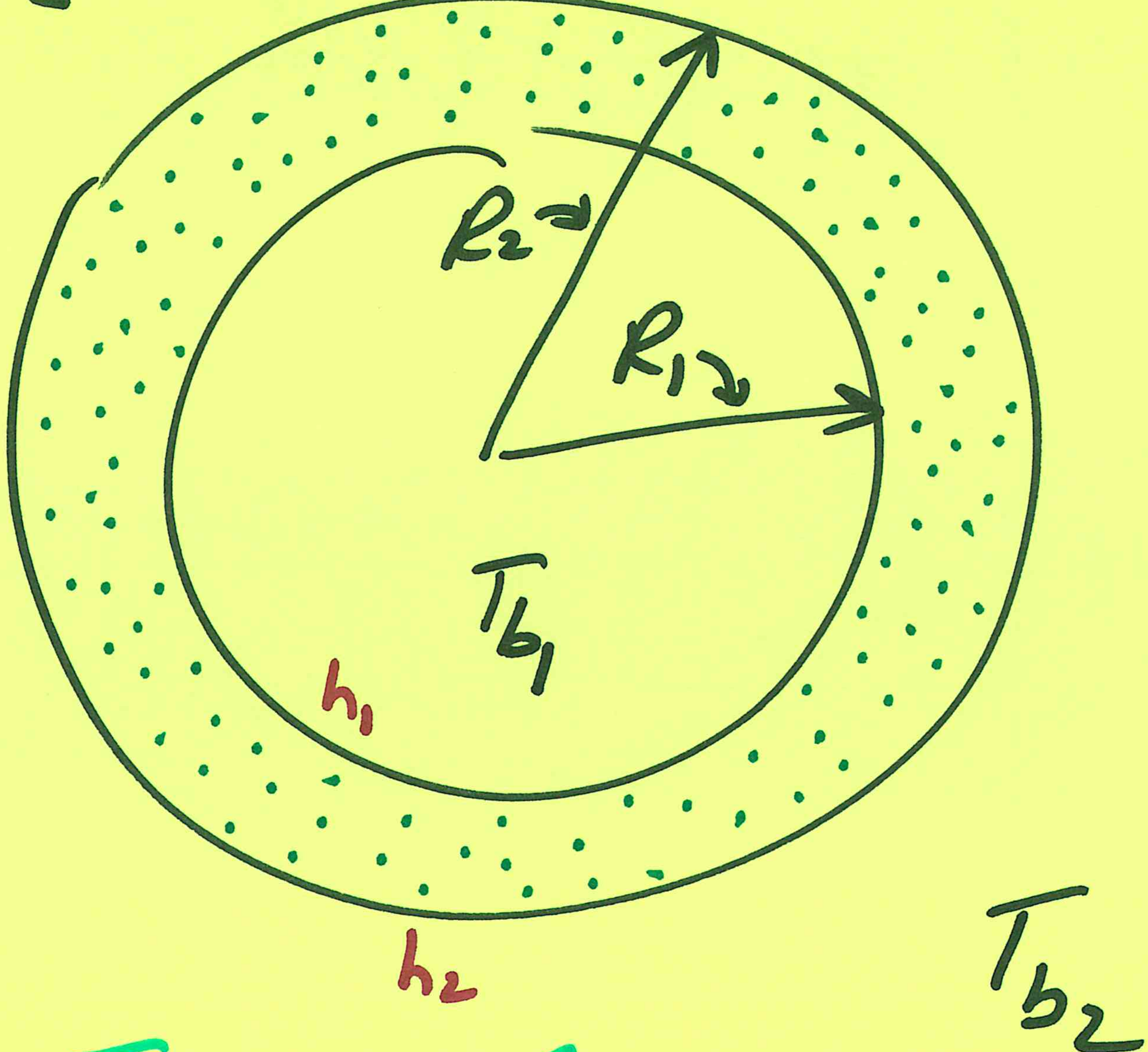
$$= \left(\frac{T_1 - T_2}{\ln R_1/R_2}\right)\left[\ln r - \ln R_2\right] + T_2$$

$$\boxed{T - T_2 = (T_1 - T_2)\left(\frac{\ln r/R_2}{\ln R_1/R_2}\right)}$$

matches slides //

EXAMPLE 4

- radial heat conduction
- $T_{b_1} > T_{b_2}$
- heat xfer coef h_1, h_2
- steady



What is Temp profile?
What is heat flux $q_r/A = ?$

Soln is the same as example 3 (8)
up to BCs:

$$T = c_1 \ln r + c_2$$

$$\frac{q_r}{A} = -k c_1 \left(\frac{1}{r}\right)$$

1D, steady, radial, long, θ symmetric, no current, no rxn

BC: Newton's Law of Cooling

$$h \left| \underbrace{T_b - T_w}_{\text{what order gives}} \right| = \left| \frac{q_r}{A} \right|$$

correct sign for flux?

★ heat goes in dir of increasing $r \Rightarrow$ flux > 0

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BC:

$$r = R_1 \qquad h_1 \underbrace{(T_{b_1} - T_{w_1})}^{\text{positive}} = \overbrace{-k c_1 \frac{1}{R_1}}^{\text{positive}} \Big\} \text{ FLUX at } R_1$$

$$r = R_2 \qquad h_2 (T_{w_2} - T_{b_2}) = -k c_1 \frac{1}{R_2} \Big\} \text{ FLUX at } R_2$$

not equal

$$T_{w_1} = T(R_1) = c_1 \ln R_1 + c_2$$

$$T_{w_2} = T(R_2) = c_1 \ln R_2 + c_2$$

4 eqns

4 unknowns

$\Rightarrow$ SOLVE for $T(r)$

SOLN (solving for $T(r)$, $\frac{q_r}{A}(r)$)

Substitute in T_{w1}, T_{w2}:

$$\left.\begin{aligned} h_1 \left(T_{b1} - c_1 \ln R_1 - c_2\right) &= -\frac{k c_1}{R_1} \\ h_2 \left(c_1 \ln R_2 + c_2\right) - h_2 T_{b2} &= -\frac{k c_1}{R_2} \end{aligned}\right\} \text{2 eqns, 2 unknowns}$$

Next, solve first eqn for c_2:

$$\boxed{c_2 = \frac{k c_1}{R_1 h_1} + T_{b1} - c_1 \ln R_1}$$

substitute this into 2nd eqn, solve c_1

SOLVE FOR c_1:

$$-\frac{k c_1}{R_2 h_2} - c_1 \ln R_2 + T_{b_2}$$

$$= C_2 = \frac{k c_1}{R_1 h_1} + T_{b_1} - c_1 \ln R_1$$

gather c_1's:

$$c_1 \left(-\frac{k}{h_2 R_2} - \frac{k}{R_1 h_1} + \ln R_1 - \ln R_2 \right)$$

$$= \left(T_{b_1} - T_{b_2} \right)$$

$$\boxed{C_1 = \frac{-\frac{1}{k}\left(T_{b_1} - T_{b_2}\right)}{\left(\frac{1}{R_1 h_1} + \frac{1}{k}\ln\frac{R_2}{R_1} + \frac{1}{R_2 h_2}\right)}}$$

substitute into C_2 eqn:

$$C_2 = T_{b_2} - C_1\left(\frac{k}{h_2 R_2} + \ln R_2\right)$$

substitute c_1, c_2 into temp profile:

$$T = c_1 \ln r + c_2$$

$$= c_1 \ln r + T_{b2} - c_1\left(\frac{k}{h_2 R_2} + \ln R_2\right)$$

$$(T - T_{b2}) = c_1\left[\ln r - \ln R_2 - \frac{k}{h_2 R_2}\right]$$

$$= c_1\left[\ln \frac{r}{R_2} - \frac{k}{h_2 R_2}\right]$$

substitute previous expression... $\rightarrow$

$$(T - T_{b_2}) = \frac{-\frac{1}{k}(T_{b_1} - T_{b_2})\left[\ln\frac{r}{R_2} - \frac{k}{h_2 R_2}\right]}{\left(\frac{1}{R_1 h_1} + \frac{1}{k}\ln\frac{R_2}{R_1} + \frac{1}{R_2 h_2}\right)}$$

$$\boxed{(T - T_{b_2}) = \frac{(T_{b_1} - T_{b_2})\left[\frac{1}{k}\ln\frac{R_2}{r} + \frac{1}{h_2 R_2}\right]}{\left(\frac{1}{R_1 h_1} + \frac{1}{k}\ln\frac{R_2}{R_1} + \frac{1}{R_2 h_2}\right)}}$$

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What is $\frac{q_r}{A}$?

substitute

$$\frac{q_r}{A} = -k\,c_1\left(\frac{1}{r}\right)$$

$$\boxed{\frac{q_r}{A} = \frac{(T_{b1} - T_{b2})\left[\frac{1}{r}\right]}{\left(\frac{1}{R_1 h_1} + \frac{1}{k}\ln\frac{R_2}{R_1} + \frac{1}{R_2 h_2}\right)}}$$

matches slides

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