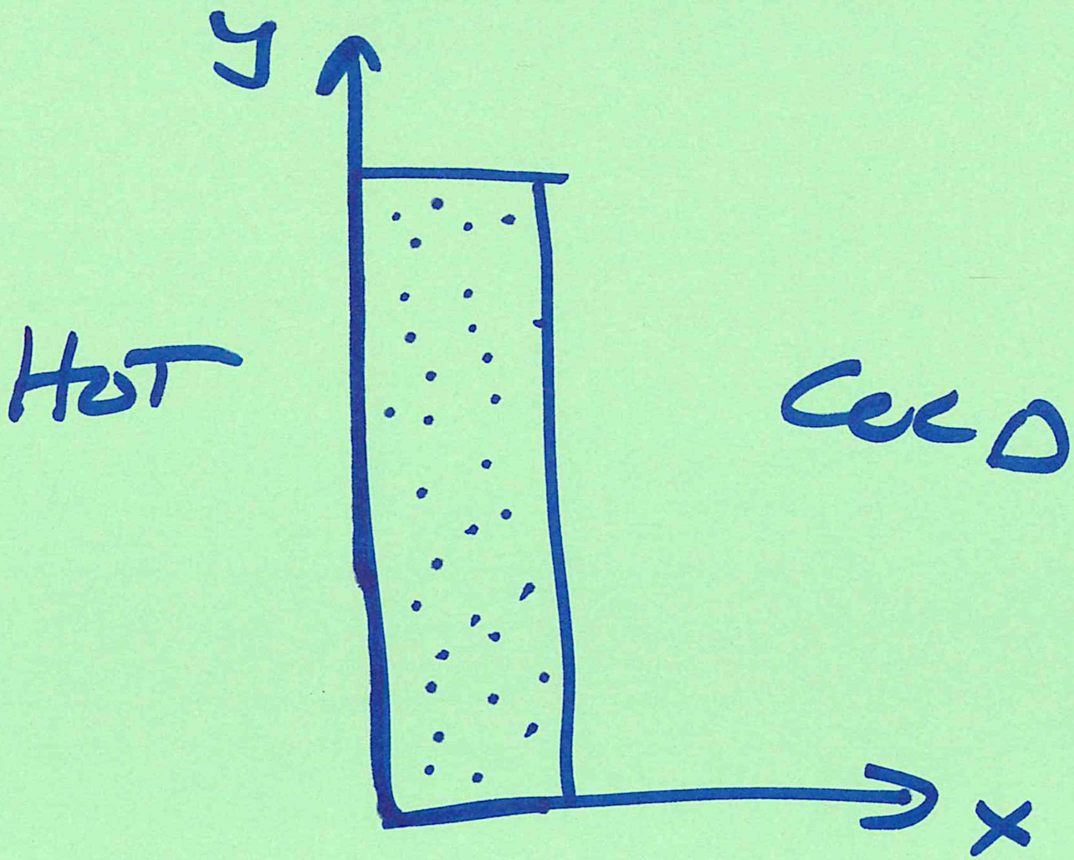
CBE 1445 10-28-19 1

1D Heat Conduction in a Slab



- steady
- wide
- tall

Micro E-BAL

(see sheet)

to match slides, use energy balance w/ $\tilde{q}$

1D, steady rectangular

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The **Equation of Energy** in Cartesian, cylindrical, and spherical coordinates for Newtonian fluids of constant density, with source term S_e. Source could be electrical energy due to current flow, chemical energy, etc. Two cases are presented: the general case where thermal conductivity may be a function of temperature (vector flux $\underset{\sim}{\tilde{q}} = \underset{\sim}{q}/area$ appears in the equations); and the more usual case, where thermal conductivity is constant.

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Microscopic energy balance, in terms of flux; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underset{\sim}{v} \cdot \nabla T \right) = -\nabla \cdot \underset{\sim}{\tilde{q}} + S_e$$

Microscopic energy balance, in terms of flux; Cartesian coordinates

$$\rho \hat{C}_p \left(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_x \frac{\partial T}{\partial x}} + \cancel{v_y \frac{\partial T}{\partial y}} + \cancel{v_z \frac{\partial T}{\partial z}} \right) = -\left(\frac{\partial \tilde{q}_x}{\partial x} + \cancel{\frac{\partial \tilde{q}_y}{\partial y}} + \cancel{\frac{\partial \tilde{q}_z}{\partial z}} \right) + \cancel{S_e}$$

no electric current

no reaction

Microscopic energy balance, in terms of flux; cylindrical coordinates

steady

solid

wide + tall

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = -\left(\frac{1}{r} \frac{\partial (r \tilde{q}_r)}{\partial r} + \frac{1}{r} \frac{\partial \tilde{q}_\theta}{\partial \theta} + \frac{\partial \tilde{q}_z}{\partial z} \right) + S_e$$

Microscopic energy balance, in terms of flux; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = -\left(\frac{1}{r^2} \frac{\partial (r^2 \tilde{q}_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\tilde{q}_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial \tilde{q}_\phi}{\partial \phi} \right) + S_e$$

Fourier's law of heat conduction, Gibbs notation: $\underset{\sim}{\tilde{q}} = -k \nabla T$

Fourier's law of heat conduction, Cartesian coordinates: $\begin{pmatrix} \tilde{q}_x \\ \tilde{q}_y \\ \tilde{q}_z \end{pmatrix}_{xyz} = \begin{pmatrix} -k \frac{\partial T}{\partial x} \\ -k \frac{\partial T}{\partial y} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{xyz}$

Fourier's law of heat conduction, cylindrical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -k \frac{\partial T}{\partial z} \end{pmatrix}_{r\theta z}$

Fourier's law of heat conduction, spherical coordinates: $\begin{pmatrix} \tilde{q}_r \\ \tilde{q}_\theta \\ \tilde{q}_\phi \end{pmatrix}_{r\theta \phi} = \begin{pmatrix} -k \frac{\partial T}{\partial r} \\ -\frac{k}{r} \frac{\partial T}{\partial \theta} \\ -\frac{k}{r \sin\theta} \frac{\partial T}{\partial \phi} \end{pmatrix}_{r\theta \phi}$

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micro e-balance:

$$0 = -\frac{d}{dx}(\tilde{q}_x)$$

$$\Rightarrow \quad \tilde{q}_x = \boxed{\frac{q_x}{A} = c_1}$$

★ FLUX IS CONSTANT

Fourier's law:

$$c_1 = \frac{q_x}{A} = -k\frac{dT}{dx}$$

★ Temp prof is linear

$$\frac{dT}{dx} = \frac{-c_1}{k} \quad \Rightarrow \quad \boxed{T = \left(\frac{-c_1}{k}\right)x + c_2}$$

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Boundary conditions:

$x=0 \quad T=T_1 \Rightarrow \boxed{c_2 = T_1}$

$x=B \quad T=T_2 \Rightarrow T_2 = \left(\frac{-c_1}{k}\right)B + T_1$

$$\boxed{c_1 = \frac{-k(T_2 - T_1)}{B}}$$

$$\boxed{\begin{array}{ll} \text{FLUX:} & \dfrac{q_x}{A} = \left(\dfrac{-k}{B}\right)(T_2 - T_1) \\ \text{TEMP PROFILE: } T(x) & T = \dfrac{(T_2 - T_1)}{B}x + T_1 \end{array}}$$

Alternatively, use the micro-ebal in terms of temperature (no $\tilde{q}$'s)

(see next page)

micro e-bal (in terms of T)

$$0 = \cancel{k}\frac{d^2T}{dx^2}$$

divide both sides by k

$$\frac{d}{dx}\underbrace{\left(\frac{dT}{dx}\right)}_{\equiv \Phi}=0$$

$$\frac{d\Phi}{dx}=0 \Rightarrow \Phi = \tilde{c}_1 = \frac{dT}{dx}$$

1D STEADY RECTANGULAR HEAT CONDUCTION ⑥

The **Equation of Energy** for systems with **constant k**

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\cancel{\frac{\partial T}{\partial t}} + \cancel{v_x} \frac{\partial T}{\partial x} + \cancel{v_y} \frac{\partial T}{\partial y} + \cancel{v_z} \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}} \right) + \cancel{S_e}$$

steady

$\underline{v} = 0$

no electrical conduction

no reaction

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right)$$
$$= k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S_e$$

Reference: F. A. Morrison, "Web Appendix to *An Introduction to Fluid Mechanics*," Cambridge University Press, New York, 2013 (*pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf*). This worksheet is on the web at *pages.mtu.edu/~fmorriso/cm310/energy.pdf*.

$$\frac{dT}{dx} = \tilde{c}_1$$

$$\boxed{T = \tilde{c}_1 x + \tilde{c}_2}$$

BC: $x = 0 \quad T = T_1 \implies \boxed{\tilde{c}_2 = T_1}$

$x = B \quad T = T_2 \implies T_2 = \tilde{c}_1 B + T_1$

$$\boxed{\tilde{c}_1 = \left(\frac{T_2 - T_1}{B}\right)}$$

flux = ?
(FOURIER'S LAW) $\quad \dfrac{q_x}{A} = -k\dfrac{dT}{dx}$

$$\boxed{\begin{array}{ll} \text{FLUX:} & \dfrac{q_x}{A} = -k\tilde{c}_1 = -k\left(\dfrac{T_2 - T_1}{B}\right) \\[2em] \text{TEMP PROFILE: } (T(x)) & T = \left(\dfrac{T_2 - T_1}{B}\right)x + T_1 \end{array}}$$

<u>SAME</u> both methods

(but c_1, c_2 + $\tilde{c}_1, \tilde{c}_2$ are not the same constants)