CM3120: Module 2

Unsteady State Heat Transfer

I. Introduction
II. Unsteady Microscopic Energy Balance—(slash and burn)
III. Unsteady Macroscopic Energy Balance
IV. Dimensional Analysis (unsteady)—Biot number, Fourier number
V. Low Biot number solutions—Lumped parameter analysis
VI. Short Cut Solutions—(initial temperature T_0; finite h), Gurney and Lurie charts (as a function of position, $m = 1/\mathrm{Bi}$, and Fo); Heissler charts (center point only, as a function of $m = 1/\mathrm{Bi}$, and Fo)
VII. Full Analytical Solutions (stretch)

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CM3120: Module 2

Module 2 Lecture VII:

Full Analytical Solutions

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www.chem.mtu.edu/~fmorriso/cm3120/cm3120.html

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We indicated that there are three ranges of Biot number to consider:

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

$$\text{Bi} = \frac{D_{char}/k}{1/h}$$

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

We have been exploring these ranges

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$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

$$\text{Bi} = \frac{D/k}{1/h}$$

Quantifies the tradeoffs between the resistance to heat flow (due to conductivity, D/k) and the resistance to heat flow at the boundary ($1/h$)

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Low Bi: high k, low h

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$$\text{Bi} = \frac{D/k}{1/h}$$

Quantifies the tradeoffs between the resistance to heat flow (due to conductivity, D/k) and the resistance to heat flow at the boundary ($1/h$)

When both processes affect the outcomes, the full solution may be necessary. For uniform starting temperatures, the solutions are published.

the body. | high h

$T = T(x, y, z, t)$

hard BC

At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is <u>uniform in a finite body</u>; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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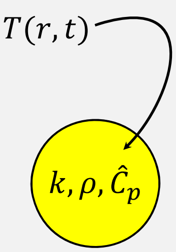
Unsteady State Heat Transfer: Analytical Solutions

No Mechanism Dominates

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (D/k) and external ($1/h$) resistances are important
- We need to match <u>*measurable*</u> quantities with <u>*calculable*</u> quantities
- ⇒ ***Microscopic*** Energy Balance
- ⇒*Uncertainty considerations*

$T = T(r, t)$



Fluid bulk temperature= T_∞

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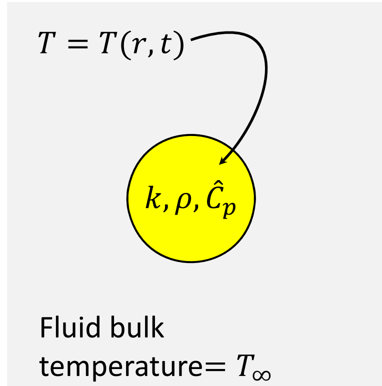
Unsteady State Heat Transfer: Analytical Solutions

No Mechanism Dominates

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (D/k) and external $(1/h)$ resistances are important
- We need to match *measurable* quantities with *calculable* quantities
- ⇒ ***Microscopic*** Energy Balance
- ⇒*Uncertainty considerations*

?



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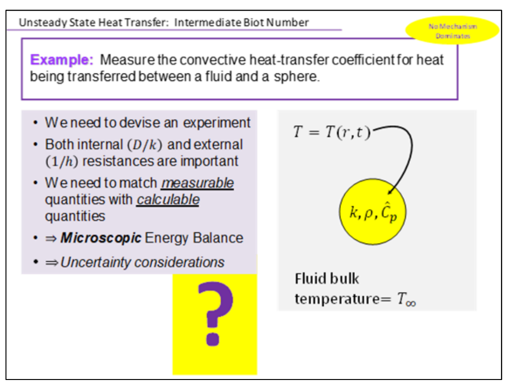

Thinking

- Create an unsteady state heat transfer situation…
- Measure . . .?
- Compare . . .?
- Consider uncertainty in measurements . . . ?

You try.

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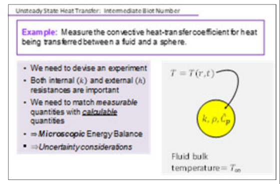
Unsteady State Heat Transfer: Analytical Solutions

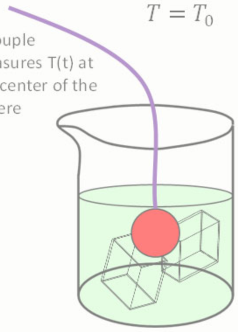


Experiment: Measure $T(t)$ at the center of a sphere $(r = 0)$:

Initially:

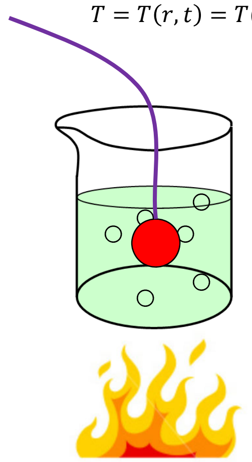
$t < t_0$
$T = T_0$

T-couple measures T(t) at the center of the sphere



Suddenly:

$t \geq t_0$
$T = T(r, t) = T(0, t)$



Unsteady state heat transfer takes place.

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Unsteady State Heat Transfer: Analytical Solutions

Experiment: Measure $T(t)$ at the center of a sphere $(r = 0)$:

Initially:

$t < t_0$
$T = T_0$

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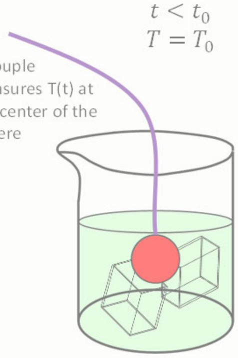

Suddenly:

$t \geq t_0$
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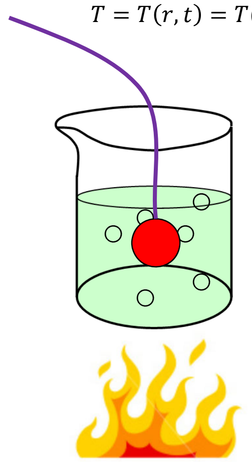

Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
2.11E-01	7.44E+00
3.09E-01	7.44E+00
4.09E-01	7.57E+00
5.24E-01	7.46E+00
6.23E-01	7.49E+00
7.39E-01	7.53E+00
8.37E-01	7.46E+00
9.54E-01	7.59E+00
1.05E+00	7.53E+00
1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
…	…

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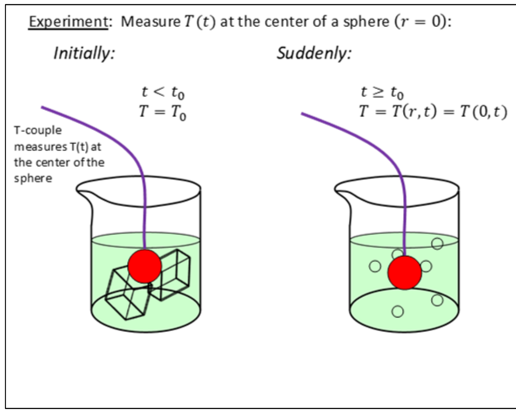
Unsteady State Heat Transfer: Analytical Solutions

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):



Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
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6.23E-01	7.49E+00
7.39E-01	7.53E+00
8.37E-01	7.46E+00
9.54E-01	7.59E+00
1.05E+00	7.53E+00
1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
...	...

Modeling

What are the modeling equations?

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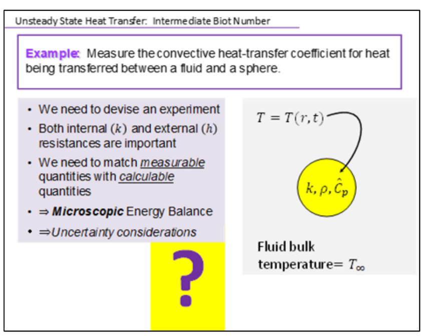

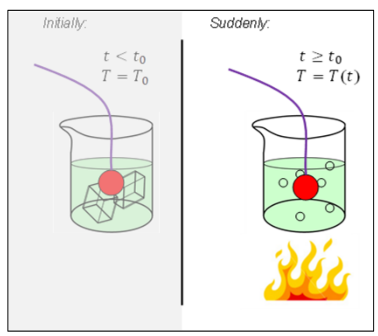

Can we meet our objective?

To determine h:

- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

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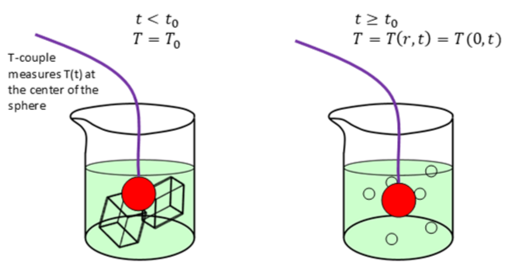
Unsteady State Heat Transfer: Analytical Solutions

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):

Initially: *Suddenly:*



Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
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1.05E+00	7.53E+00
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...	...

Modeling

What are the modeling equations?

You try.

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Unsteady State Heat Transfer: Analytical Solutions

Microscopic Energy Balance

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial T}{\partial \phi} \right) = k \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S$$

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

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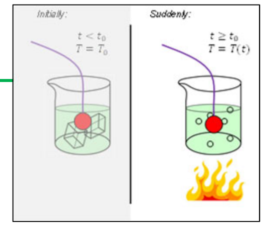
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$$\frac{\partial T}{\partial t} = \underbrace{\frac{k}{\rho \hat{C}_p}}_{\equiv \alpha} \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right)$$

- Unsteady
- Solid ($\underline{v} = 0$)
- θ, ϕ symmetry
- No current, no rxn

Boundary conditions:

$r = R,\quad \frac{q_r}{A} = -k\frac{\partial T}{\partial r} = h(T(r) - T_{bulk})\quad t > 0$

$r = 0,\quad \frac{q_r}{A} = -k\frac{\partial T}{\partial r} = 0\quad \forall t$

Initial condition:

$t = 0,\quad T = T_{initial}\quad \forall r$

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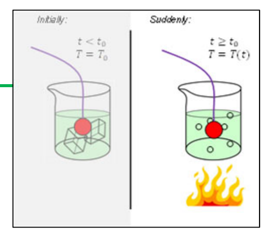
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

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("∀" means "for all")

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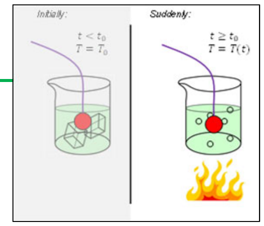
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$\equiv \alpha$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right)$$

- Unsteady
- Solid ($\underline{v} = 0$)

Bc

Now, Solve

$r = 0,$ $\frac{q_r}{A} = -k\frac{\partial T}{\partial r} = 0$ $\forall t$

Initial condition:

$t = 0,$ $T = T_{initial}$ $\forall r$

("∀" means "for all")

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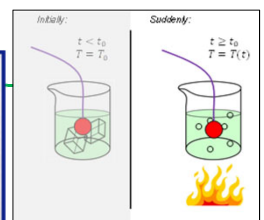
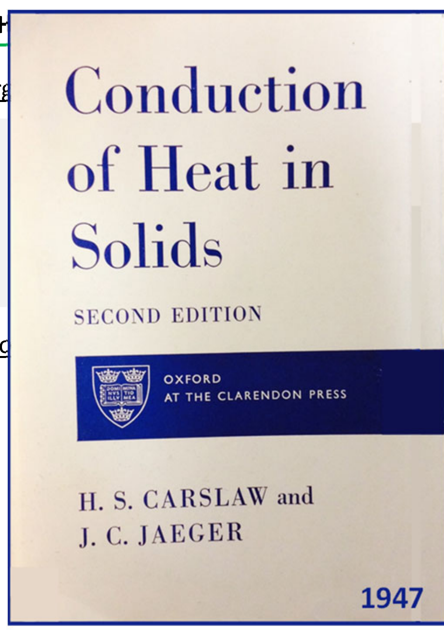
Unsteady State H



Microscopic energ



- nsteady
- lid ($\underline{v} = 0$)
- ϕ symmetry
- current, no rxn

Boundary conditic

$r = R,$ > 0

$r = 0,$ $\forall t$

Initial condition:

$t = 0,$ $\forall r$

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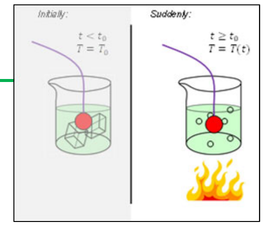
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

(Carslaw and Yeager, 1959, eqn 10, p238)
Incropera and DeWitt, 7th ed, eqn 5.51a, p303

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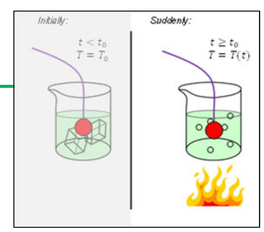
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

Note that $D_{char} = R$ for this solution

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
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$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

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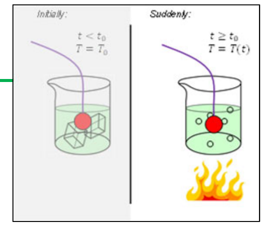
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

Depends on material ($\alpha = k/\rho \hat{C}_p$), and heat transfer processes at surface (h)

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

We're interested in $T(r,t)$ at the center of the sphere, $r = 0$.

(Carslaw and Yeager, 1959, eqn 10, p238)
Incropera and DeWitt, 7th ed, eqn 5.51a, p303

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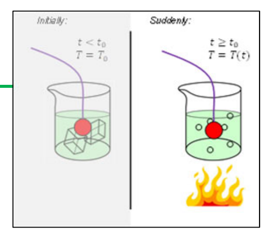
Unsteady State Heat Transfer to a Sphere



What does ***this*** look like?

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

Let's plot it to find out. (Excel)

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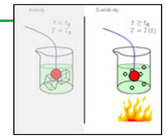
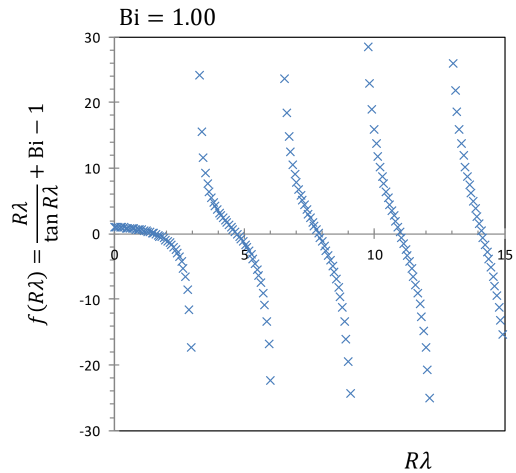
Unsteady State Heat Transfer to a Sphere



$$\text{Bi} \equiv \frac{hR}{k}$$

Eigenvalues are the roots of the characteristic equation



Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1$$

- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

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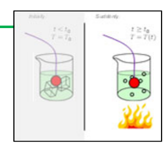
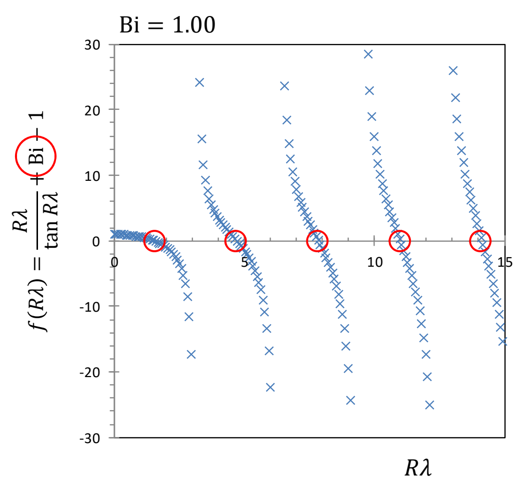
Unsteady State Heat Transfer to a Sphere



$$\text{Bi} \equiv \frac{hR}{k}$$

Eigenvalues are the roots of the characteristic equation



Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1$$

- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

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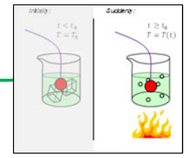
Let's plot it to find out: what are the variables?

$\text{Bi} \equiv \frac{hR}{k}$ $\quad \text{Fo} \equiv \frac{\alpha t}{R^2}$



Solution:

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

Exponential decay with Fo (scaled time)

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2} \left(\begin{matrix}\text{bunch of terms} \\ \text{that vary with Bi and } \lambda_n(\text{Bi})\end{matrix}\right)$$

$\lambda_n(\text{Bi})$ varies only with Bi and n:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

Characteristic Equation

If we choose a fixed Bi, then ξ only varies with Fo

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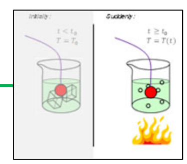
If we choose a fixed Bi, then ξ only varies with Fo

$\text{Bi} \equiv \frac{hR}{k}$ $\quad \text{Fo} \equiv \frac{\alpha t}{R^2}$



For a **fixed Bi:**

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

Exponential decay with Fo (scaled time)

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2} \left(\begin{matrix}\text{bunch of terms} \\ \text{that vary with Bi and } \lambda_n(\text{Bi})\end{matrix}\right)$$

An infinite sum of decaying **exponentials**

- whose *argument* is Fourier number scaled by something that depends on Biot number and n
- with a prefactor that also depends on Biot number and n

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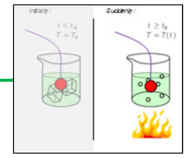
If we choose a **fixed Bi**, then ξ only varies with Fo

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$



For a **fixed Bi:**

$$\xi(0, \text{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 Fo}$$

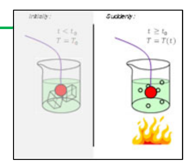
An infinite sum of decaying **exponentials**

- $\tilde{C}_n$ depends on n through λ_n
- λ_n are calculated (numerically) from the roots of this equation:

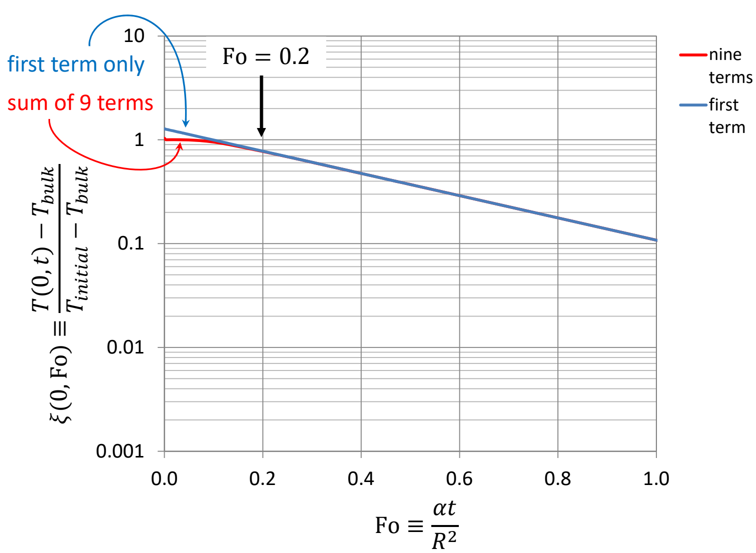
$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Let's plot $\xi(0, \text{Fo})$

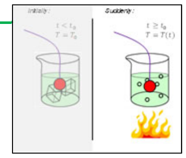
Unsteady State Heat Transfer to a Sphere



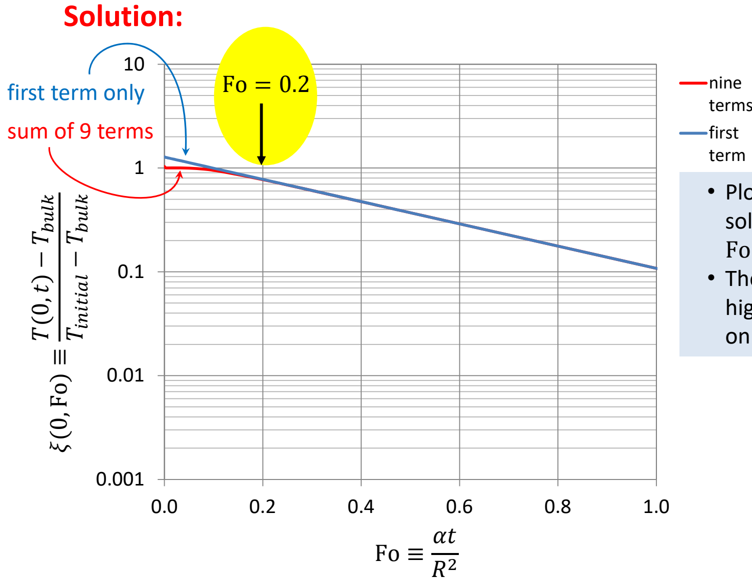
Solution:



Unsteady State Heat Transfer to a Sphere



Solution:



- Plotted log-linear, the solution is linear for $\text{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$\text{Bi} \equiv \frac{hR}{k}$

$\frac{1}{\text{Bi}} = \frac{k}{hR}$

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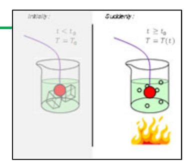
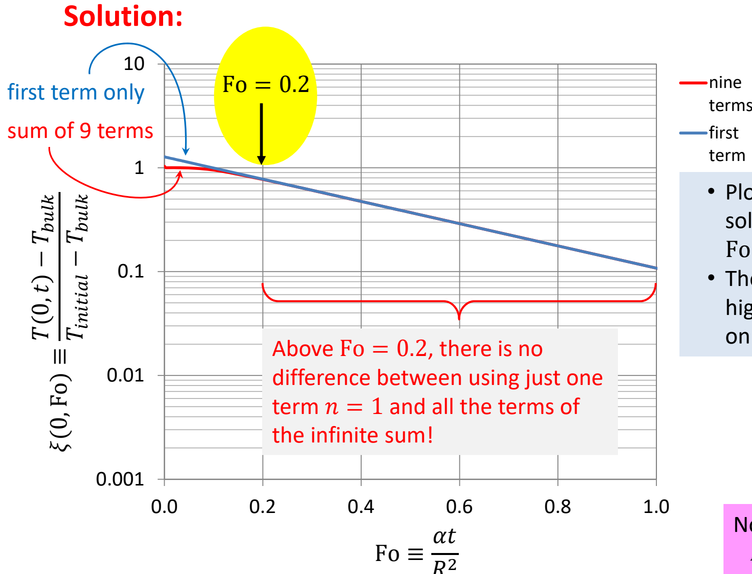
Unsteady State Heat Transfer to a Sphere



Solution:



- Plotted log-linear, the solution is linear for $\text{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$\text{Bi} \equiv \frac{hR}{k}$

$\frac{1}{\text{Bi}} = \frac{k}{hR}$

Note: $D_{char} = R$ in this case

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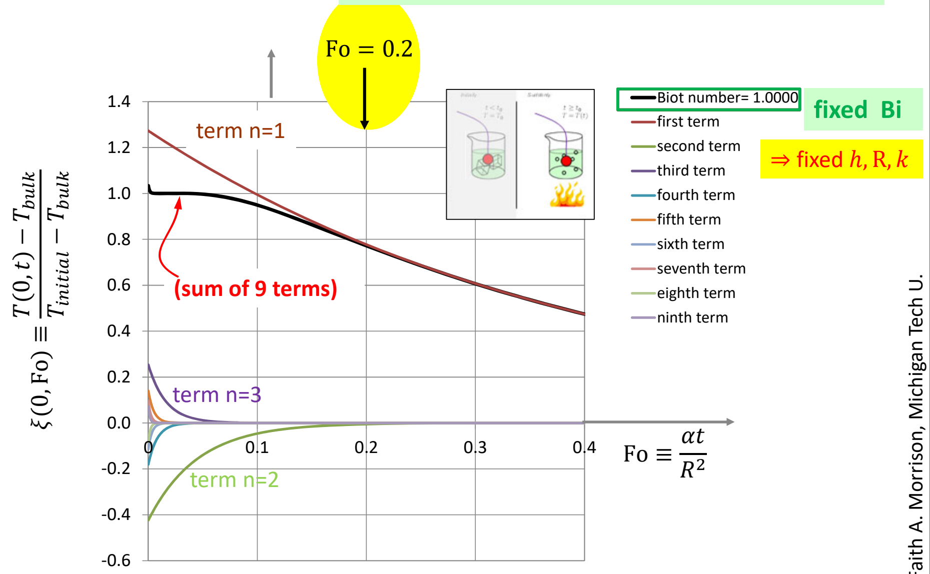
Unsteady State Heat Transfer to a Sphere

What are all those higher order terms contributing to the solution?



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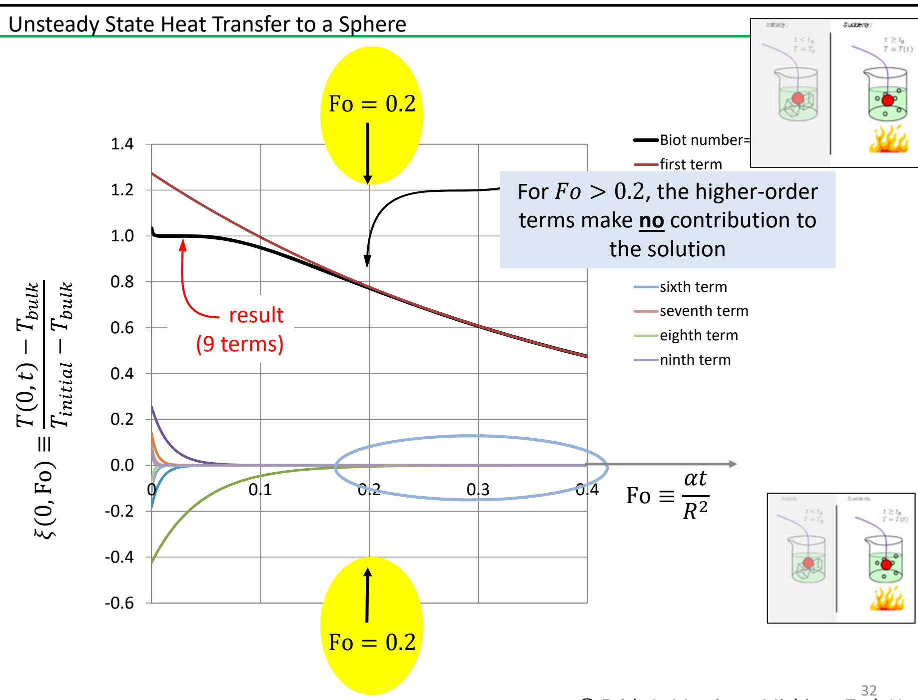
Unsteady State Heat Transfer to a Sphere



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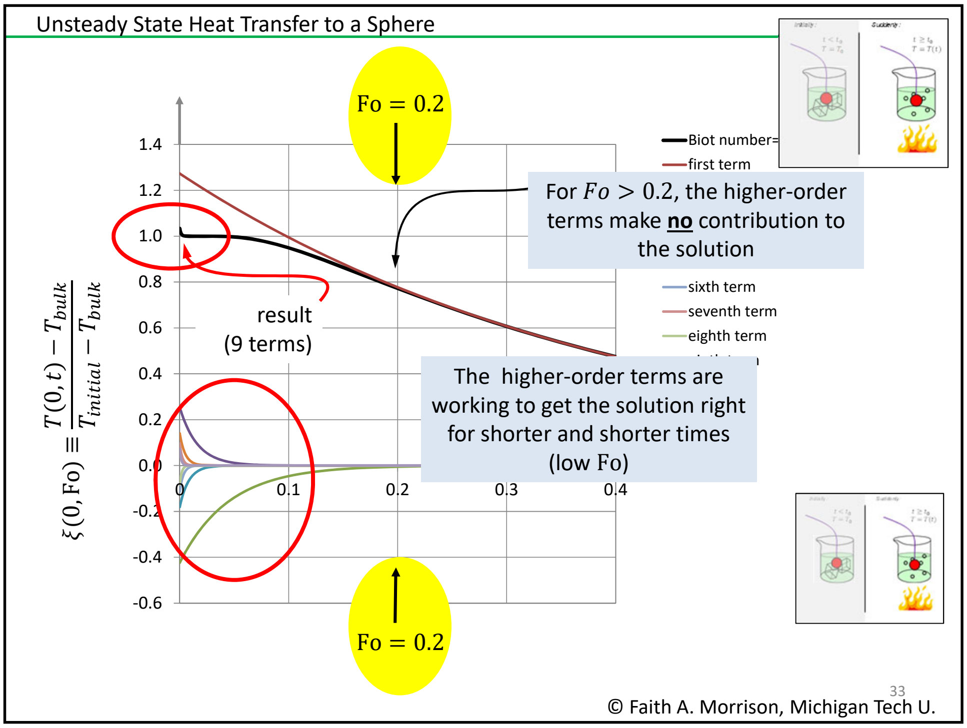
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
result (9 terms)
The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)
$\xi(0,\text{Fo}) \equiv \frac{T(0,t)-T_{bulk}}{T_{initial}-T_{bulk}}$
Biot number=
first term
sixth term
seventh term
eighth term
Fo = 0.2
33
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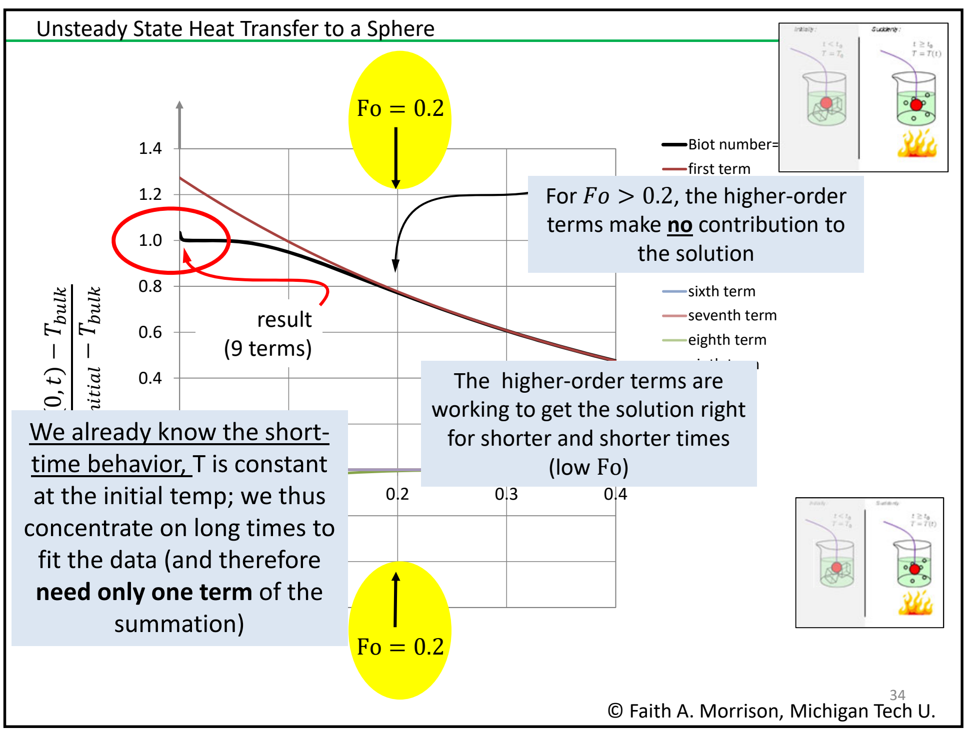
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
result (9 terms)
The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)
We already know the short-time behavior, T is constant at the initial temp; we thus concentrate on long times to fit the data (and therefore **need only one term** of the summation)
Biot number=
first term
sixth term
seventh term
eighth term
Fo = 0.2
34
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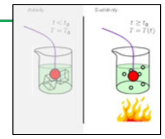
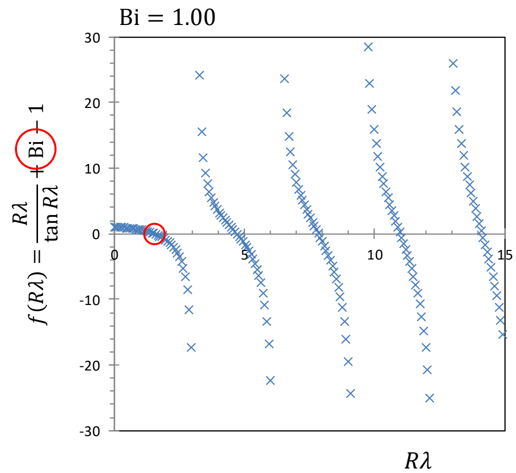
Unsteady State Heat Transfer to a Sphere



$$\mathrm{Bi} \equiv \frac{hR}{k}$$

Eigenvalues are the roots of the characteristic equation

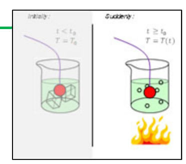


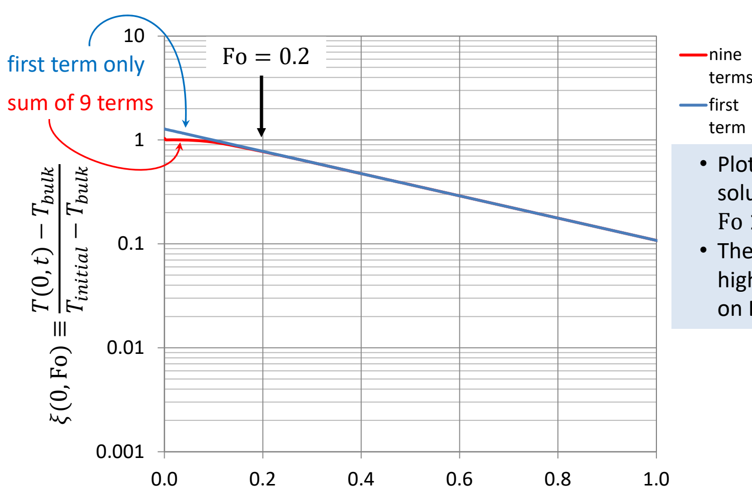
Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1$$

- The λ_1 eigenvalue is the one that dominates at long time
- The value of λ_1 depends on Biot number Bi

Unsteady State Heat Transfer to a Sphere





$$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$$

- Plotted log-linear, the solution is linear for $\mathrm{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$$\mathrm{Bi} \equiv \frac{hR}{k}$$

$$\frac{1}{\mathrm{Bi}} = \frac{k}{hR}$$

Unsteady State Heat Transfer to a Sphere

No Mechanism Dominates

$$\text{Bi} \equiv \frac{hR}{k}$$

$$\text{Fo} \equiv \frac{\alpha t}{R^2}$$

Summary

- For a fixed Bi the results are only a function of Fo.
- Solution is an infinite sum of terms.
- Each term corresponds to one eigenvalue, λ_n
- The first term $n = 1$ (λ_1) is the dominant term
- The $n > 1$ terms alternate in sign (positive and negative)
- Higher terms are “fixing” the short time behavior
- At fixed Biot number, the time-dependence is an exponential decay (for $\text{Fo} > 0.2$); this is linear on a log-linear plot versus Fo

So, actually, it turns out all we need are those slopes as a function of Biot number.

Question: How do various values of Biot number affect the heat transfer that occurs?

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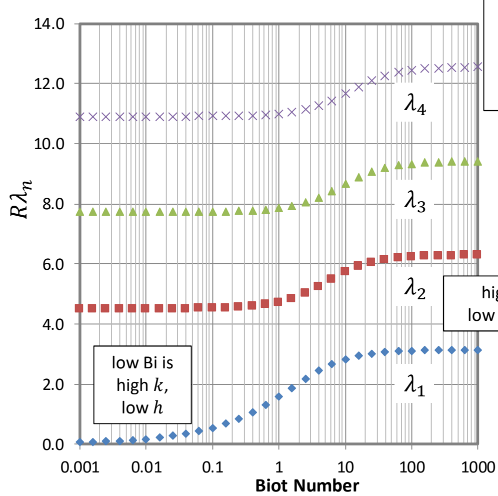
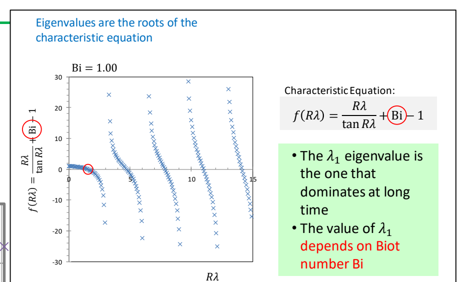
Unsteady State Heat Transfer to a Sphere

$$\text{Bi} \equiv \frac{hR}{k}$$





- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low ($\text{Bi} < 0.1$) and high Bi ($\text{Bi} > 10$) the solution becomes independent of Bi

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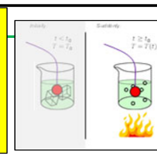
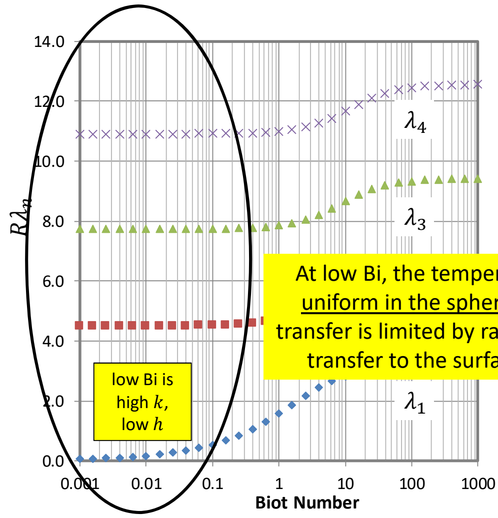
Unsteady State Heat Transfer to a Sphere

Low Bi:
high k,
low h



$$\text{Bi} \equiv \frac{hR}{k}$$



At low Bi, the temperature is uniform in the sphere; heat transfer is limited by rate of heat transfer to the surface (h).

Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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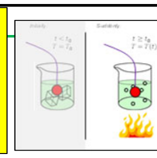
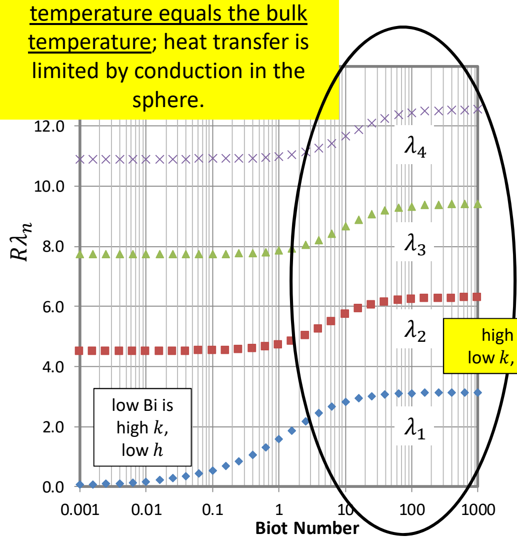
Unsteady State Heat Transfer to a Sphere

High Bi:
low k,
high h



At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the sphere.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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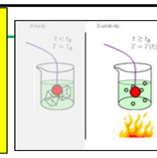
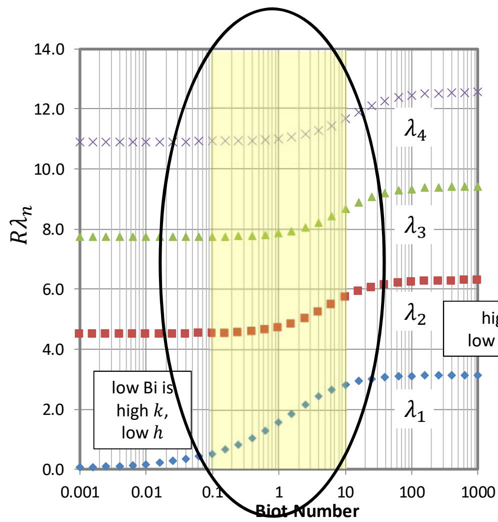
Unsteady State Heat Transfer to a Sphere

Moderate Bi: nether process dominates



At moderate Bi, heat transfer is affected by both conduction in the sphere and the rate of heat transfer to the surface.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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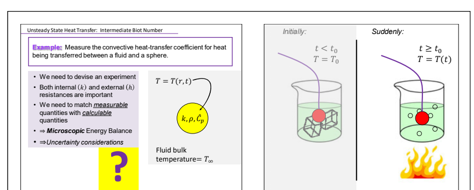
What were we trying to do?

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

Where are we in the process?

✓ We have the model

- We need the measured center-point temperature as a function of time
- We need to compare the two to deduce h.



Can we meet our objective?

To determine h:
- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

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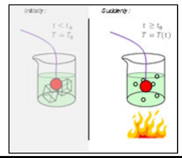
The solution of the model:

$$\xi(0, \text{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 Fo}$$

$\text{Fo} \equiv \frac{\alpha t}{R^2}$

$\text{Bi} \equiv \frac{hR}{k}$



Use to interpret data.

For a **fixed Bi, Fo > 0.2:**

$$\xi(0, \text{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 Fo}$$

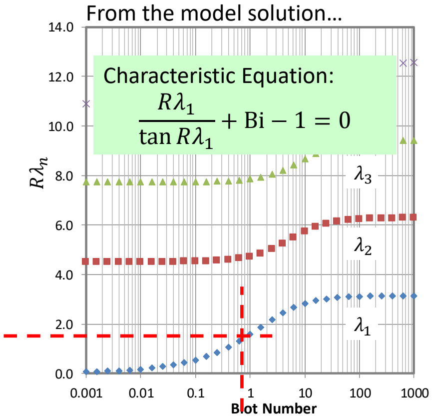
$$\ln \xi(0, \text{Fo}) = \ln(\tilde{C}_1) - \lambda_1^2 R^2 \text{Fo}$$

From experiments…

Plot: $\ln \xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> slope $= -\lambda_1^2 R^2$

From the model solution…



Once we know Bi, we can calculate h from Bi

$\text{Bi} \equiv \frac{hR}{k}$

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Experimental Data

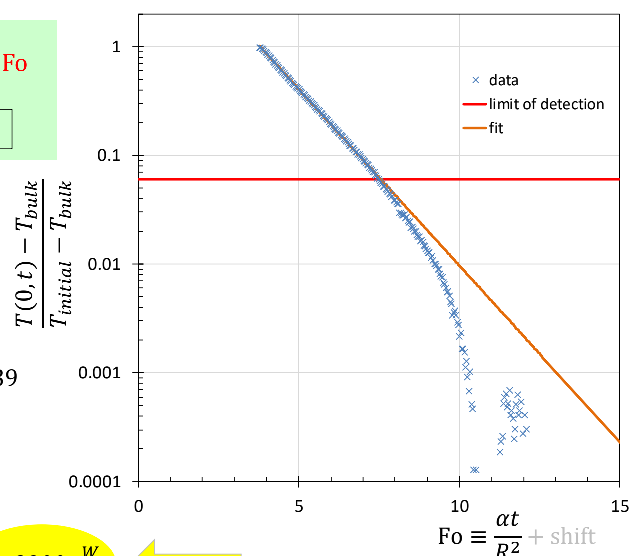
$k = 109 \frac{W}{mK}$

$R = 0.0127\ m$

From experiments…

Plot: $\ln \xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> slope $= -\lambda_1^2 R^2$



Slope$= -0.74656$

$\lambda_1 R = \sqrt{0.74656} = 0.864039$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

$\Rightarrow \text{Bi} = \frac{hR}{k} = 0.2621919,\ h = 2300 \frac{W}{m^2 K}$

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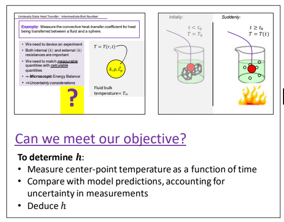
Recap:

Measure convective heat-transfer coefficient h for heat being transferred between a fluid and a sphere.

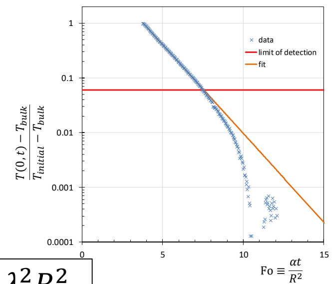
What was the process?

1. Create the scenario and the model of the scenario



2. Take data of center-point $T(t)$

3. Plot the data in a way that we can match it to the model to deduce Bi



$$\text{slope} = -\lambda_1^2 R^2$$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

$$\text{Bi} \equiv \frac{hR}{k} \Rightarrow h$$

4. Calculate h.

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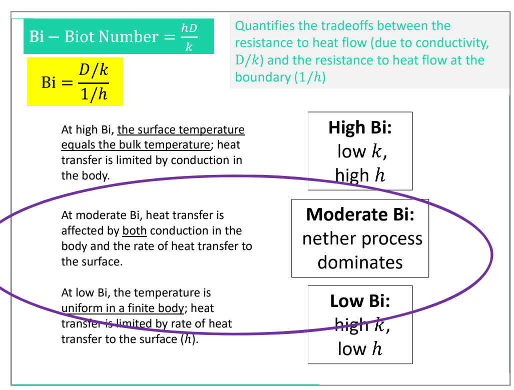
Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\text{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

Moderate Bi: The limits for "moderate" are $0.1 \leq \text{Bi} \leq 10$. When Bi is in this range, a more complete solution may be necessary; $\text{Bi} = hD_{char}/k$. (D_{char} varies with the problem)

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\text{Bi} = hV/kA < 0.1$



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Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\mathrm{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

Moderate Bi: The limits for "moderate" are $0.1 \le \mathrm{Bi} \le 10$. When Bi is in this range, a more complete solution may be necessary; $\mathrm{Bi} = hD_{char}/k$. ($D_{char}$ varies with the problem)

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\mathrm{Bi} = hV/kA < 0.1$

Bi – Biot Number $= \frac{hD}{k}$

$\mathrm{Bi} = \frac{D/k}{1/h}$

Quantifies the tradeoffs between the resistance to heat flow (due to conductivity, D/k) and the resistance to heat flow at the boundary ($1/h$)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

D_{char}=characteristic length scale

We use $D_{LP} = V/A$ only for the lumped parameter analysis. We use different D_{char} in other cases.

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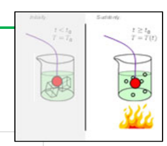
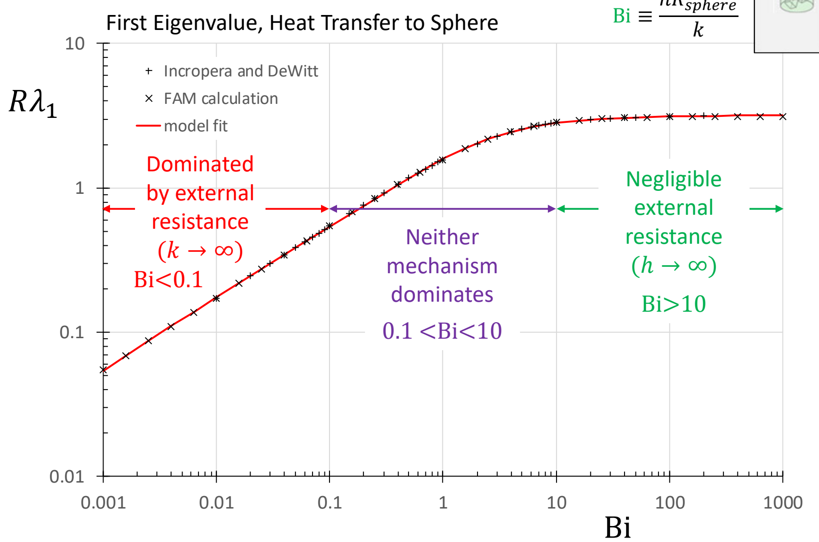
Unsteady State Heat Transfer to a Sphere



$\mathrm{Bi} \equiv \frac{hR_{sphere}}{k}$



Fit by Faith A. Morrison 31 Jan 2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

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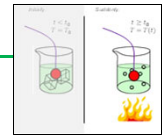
Unsteady State Heat Transfer to a Sphere



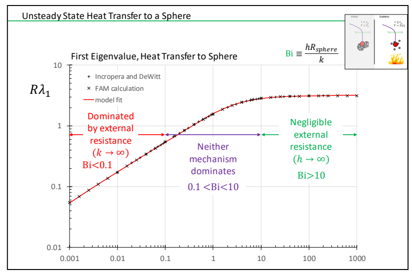
If we know $R\lambda_n$ and we're determining Bi (i.e. h), we use the characteristic equation directly.

If we know h (and hence, Bi) we need to find $\lambda_1 R$ from an iterative solution of the characteristic equation.

Or use a table or correlation for the calculated roots.

Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \mathrm{Bi} - 1 = 0$$



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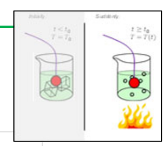
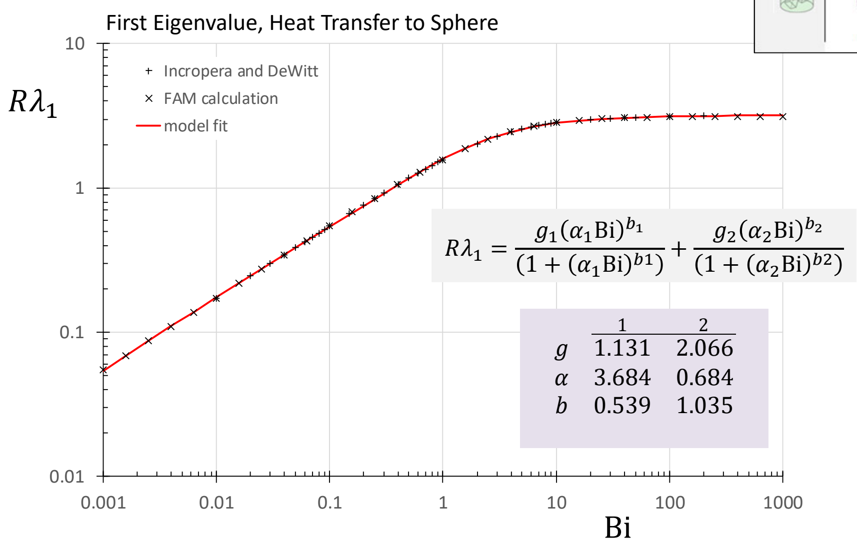
Unsteady State Heat Transfer to a Sphere





$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b_1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b_2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

Fit by Faith A. Morrison 1/31/2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

(matches true within 1.2%)

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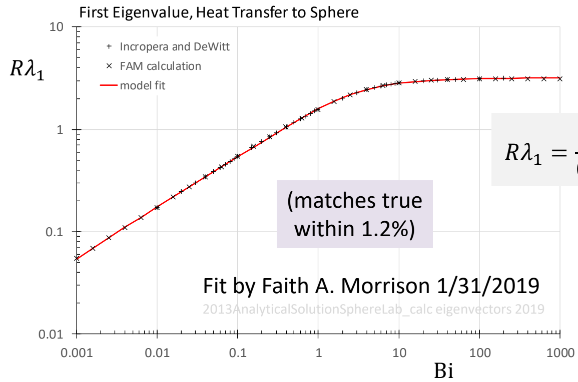
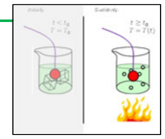
Unsteady State Heat Transfer to a Sphere





$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

For a **fixed Bi**, $\mathrm{Fo} > \mathbf{0.2}$:

$$\xi(0,\mathrm{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 Fo}$$

$$\frac{2\mathrm{Bi}[(R\lambda_1)^2 + (\mathrm{Bi}-1)^2]}{(R\lambda_1)^2 + \mathrm{Bi}(\mathrm{Bi}-1)}\left(\frac{\sin R\lambda_1}{R\lambda_1}\right) = \tilde{C}_1 = \frac{4[\sin(\lambda_1 R) - \lambda_1 R\cos(\lambda_1 R)]}{2\lambda_1 R - \sin 2\lambda_1 R}$$

The two expressions may be shown to be equivalent by substituting the characteristic equation; see FAM notebook v.53 2019; the second version is more robust due to avoiding multiplying by zero when $R\lambda_1 \to \pi$.

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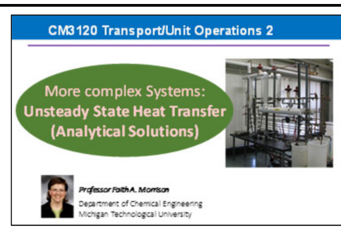
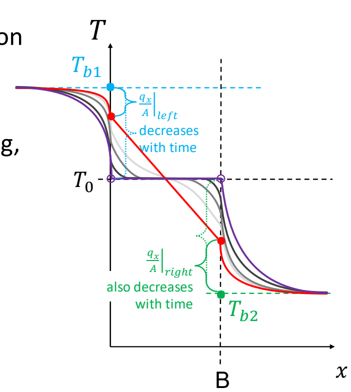
Unsteady State Heat Transfer



Summary

- Unsteady state heat transfer is very common in the chemical process industries
- Temperature distributions depend strongly on what initiates the heat transfer (usually something at the boundary)
- ***Internal resistance*** (D_{char}/k) can be limiting, irrelevant, or one among many resistances
- ***External resistance*** $(1/h)$ can be limiting irrelevant, or one among many resistances
- **Dimensional analysis**, once again, organizes the impacts of various influences $(\mathrm{Bi}, \mathrm{Fo})$



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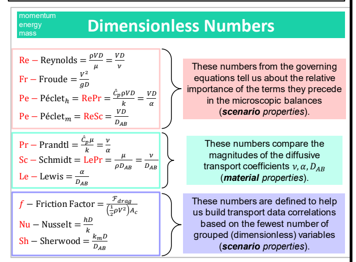
Unsteady State Heat Transfer



Summary *(continued)*

- In transport phenomena, we have dimensionless numbers that represent three important aspects of situations that interest us:
  1. The relative importance of individual terms in the equations of change
  2. The relative magnitudes of the diffusive transport coefficients ν, α, D_{AB}
  3. Scaled values of quantities of interest, e.g. wall forces, heat transfer coefficients, and mass transfer coefficients (data correlations)

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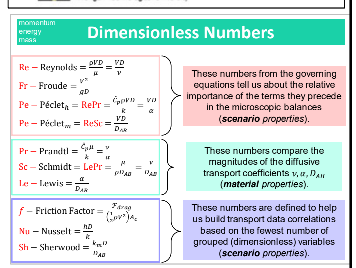
Unsteady State Heat Transfer

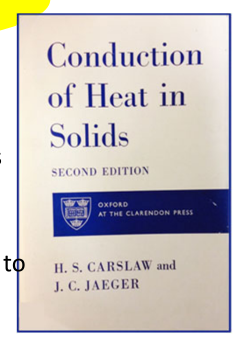


Summary *(continued)*

- If we can develop a model situation for questions of interest, the solutions of the models are often in the literature

Our responsibility in 21st century:

- Learn to develop models that will allow us to estimate or determine answers to the questions that interest us
- Learn to use published solutions (tables, charts) to answer questions that interest us



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CM3120: Module 2

✓ **Unsteady State Heat Transfer**

I. Introduction
II. Unsteady Microscopic Energy Balance—(slash and burn)
III. Unsteady Macroscopic Energy Balance
IV. Dimensional Analysis (unsteady)—Biot number, Fourier number
V. Low Biot number solutions—Lumped parameter analysis
VI. Short Cut Solutions—(initial temperature T_0; finite h), Gurney and Lurie charts (as a function of position, $m = \frac{1}{\mathrm{Bi}}$, and Fo); Heissler charts (center point only, as a function of $m = 1/\mathrm{Bi}$, and Fo)
VII. Full Analytical Solutions (stretch)

Module 2 DONE!

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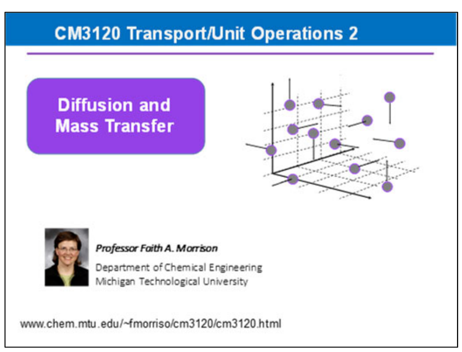
NEXT: Module 3 Diffusion & Mass Transfer



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