X. *Initial temperature f(r). Zero surface temperature*

Fourier's result has been given in (3). The solution useful for small values of $\kappa t/a^2$ is

$$v = \frac{1}{2r(\pi\kappa t)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \int_0^a r' f(r') \left\{\exp\left[-\frac{(2na+r'-r)^2}{4\kappa t}\right] - \exp\left[-\frac{(2na+r'+r)^2}{4\kappa t}\right]\right\} dr'. \tag{21}$$

XI. *Zero surface temperature. Initial temperature*†

$$f(r) = b_0 + br + cr^2 + dr^3 + \dots . \tag{22}$$

$$v = \frac{2}{ar}\sum_{n=1}^{\infty} \sin\frac{n\pi r}{a}\left\{\frac{b_0 a^2}{n\pi}(-1)^{n+1} + \frac{ba^3}{n^3\pi^3}[(n^2\pi^2-2)(-1)^{n+1}-2] + \right.$$
$$\left. + \frac{ca^4}{n^4\pi^4}(n^3\pi^3-6n\pi)(-1)^{n+1} + \frac{da^5}{n^5\pi^5}[24-(n^4\pi^4-12n^2\pi^2+24)(-1)^n] + \dots\right\} \times$$
$$\times e^{-\kappa n^2\pi^2 t/a^2}. \tag{23}$$

9.4. The sphere $0 \leqslant r < a$. Initial temperature $f(r)$. Radiation at the surface

Heat xfer to sphere w/ Newton's Law of Cooling BC

If the sphere radiates into medium at zero the equations for v are

$$\frac{\partial v}{\partial t} = \kappa\left(\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right), \quad 0 \leqslant r < a, \tag{1}$$

$$\frac{\partial v}{\partial r} + hv = 0, \qquad \text{when } r = a, \tag{2}$$

and

$$v = f(r), \quad \text{when } t = 0. \tag{3}$$

Putting $u = vr$, we have

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2}, \quad 0 < r < a, \tag{4}$$

$$u = 0, \qquad \text{when } r = 0, \tag{5}$$

$$\frac{\partial u}{\partial r} + \left(h - \frac{1}{a}\right)u = 0, \qquad \text{when } r = a, \tag{6}$$

and

$$u = rf(r), \quad \text{when } t = 0. \tag{7}$$

The problem is thus reduced to that of linear flow of heat in a slab, one end being kept at zero temperature, while at the other end radiation takes place into a medium at zero. This problem has been solved in 3.10 (11), and in this we have only to replace l by a, x by r, and h by $(ah-1)/a$. Thus the solution of (1)–(3) is‡

$$v = \frac{2}{ar}\sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t}\frac{a^2\alpha_n^2+(ah-1)^2}{a^2\alpha_n^2+ah(ah-1)}\sin\alpha_n r\int_0^a r' f(r')\sin\alpha_n r'\, dr', \tag{8}$$

† van Orstrand, *Geophysics*, 5 (1940) 57–79. He considers two more powers in the series (22), and gives numerical values for the coefficients of the first forty terms of the series (23).

‡ This solution is easily obtained directly, cf. *C.H.*, § 65. See also § 14.7 II.

where $\pm\alpha_n$, $n = 1, 2,...$ are the roots of

$$a\alpha \cot a\alpha + ah - 1 = 0. \tag{9}$$

The equation (9) is simply the equation 3.10 (7) which has already been discussed and whose roots are tabulated in Appendix IV, except that the parameter ah, which was always positive in § 3.11, is replaced by $ah-1$ which may be negative. Provided $h > 0$, i.e. $ah-1 > -1$, the remarks of §§ 3.10, 3.11 hold, and the roots of (9) are all real.†

If the initial temperature $f(r)$ is V, constant, (8) becomes‡

$$v = \frac{2hV}{r} \sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t} \frac{a^2\alpha_n^2 + (ah-1)^2}{\alpha_n^2[a^2\alpha_n^2 + ah(ah-1)]} \sin a\alpha_n \sin r\alpha_n. \tag{10}$$

If the sphere has zero initial temperature and is heated by radiation from medium at temperature kt, the solution is

$$v = k\left\{t + \frac{r^2 ah - a^2(2+ah)}{6\kappa ah}\right\} + \frac{2a^2hk}{\kappa r} \sum_{n=1}^{\infty} \frac{\sin r\alpha_n}{\alpha_n^2\{a^2\alpha_n^2 + ah(ah-1)\}\sin a\alpha_n} e^{-\kappa\alpha_n^2 t}, \tag{11}$$

where the α_n are the positive roots of (9).

If the sphere has zero initial temperature and is heated by radiation from medium at temperature $V\sin(\omega t+\epsilon)$, the temperature is

$$v = \frac{2ah\kappa V}{r} \sum_{n=1}^{\infty} \frac{\alpha_n(\kappa\alpha_n^2 \sin\epsilon - \omega\cos\epsilon)(ah-1)\sin r\alpha_n}{(\kappa^2\alpha_n^4 + \omega^2)[a^2\alpha_n^2 + ah(ah-1)]\cos a\alpha_n} e^{-\kappa\alpha_n^2 t} + \\ + \frac{a^2hVA_1}{rA_2} \sin(\omega t + \epsilon + \phi_1 - \phi_2), \tag{12}$$

where

$$A_1 e^{i\phi_1} = \sinh\omega' r \cos\omega' r + i\cosh\omega' r \sin\omega' r,$$
$$A_2 e^{i\phi_2} = a\omega'(1+i)\cosh a\omega'(1+i) + (ah-1)\sinh a\omega'(1+i),$$
$$\omega' = \sqrt{(\omega/2\kappa)},$$

and the α_n are the positive roots of (9).

9.5. Application to the determination of the conductivities of poor conductors

The expression we have just obtained for the temperature in a sphere cooling by radiation at the surface converges so rapidly that when a sufficient time has passed the terms after the first may be neglected.

† If $h < 0$ there is a pair of imaginary roots, but this case is, as always, excluded on physical grounds. If $h = 0$, that is, no flow of heat at the surface, (9) has a zero root, and a term

$$\frac{3}{a^3}\int\limits_0^a r^2 f(r)\,dr$$

has to be added to the value of (8) with $h = 0$. Cf. 3.4 (6), 7.8 (3).

‡ Surface and centre temperatures for this case are plotted against ah for various values of $\kappa t/a^2$ by Schack, *Stahl u. Eisen*, **50** (1930) 1290. See also Heisler, *Trans. Amer. Soc. Mech. Engrs.* **69** (1947) 227–36.

CONDUCTION
OF
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