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Using the Hagen-Poiseuille equation, which is only correct for laminar flow, greatly underestimates the frictional drag present in pipes. The true estimate of flow rate in household pipes (according to the internet it is around 20 *gal/min*) is two orders of magnitude lower than what we calculated in the laminar example. The large discrepancy between the laminar prediction and what is observed is evidence that the flow in household pipes is not laminar. To correctly solve the burst-pipe problem we need to know more about turbulent flow.

Turbulent flow can not be solved directly by following the microscopic momentum balance from start to finish as we did for laminar flow, but we can modify our approach by incorporating experimental observations and arrive at important results for turbulent flow, including the flow-rate/pressure-drop relationship. We pursue turbulent flow modeling in the next section.

There are important flows for which the laminar-flow solution and the Hagen-Poiseuille equation are appropriate, such as in glass-tube viscometers, which are employed to find the viscosity of fluids. We show how the Hagen-Poiseuille equation applies to such instruments in the example below.

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**EXAMPLE** *How does the measurement of efflux time  $\Delta t$  in a Cannon-Fenske routine viscometer (Figure 8.11) allow us to deduce the viscosity of the fluid?*

**SOLUTION** The Cannon-Fenske viscometer is a glass apparatus that has two fluid reservoirs connected by a tilted capillary tube (Figure 8.11). The capillary tube is manufactured to be very straight and of uniform inside diameter. An appropriate volume is charged to the lower reservoir, and, after equilibration at constant temperature, this fluid is drawn up through the capillary to fill the second reservoir to overfull. The fluid level is then allowed to drop under the pull of gravity, and the time it takes for the fluid meniscus to pass between the two marks shown in Figure 8.11 is the efflux time  $\Delta t$ .

The flow through the capillary may be analyzed as shown in Figure 8.12. Fluid in the amount that fits in the upper reservoir, volume  $\Delta V$ , flows through a capillary of length  $L$ . The time it takes for that fluid to pass through the capillary is the efflux time  $\Delta t$ . Thus, the measured flow rate  $Q$  through the capillary is  $Q = \Delta V / \Delta t$ .

$$\begin{array}{l} \text{Measured flow rate} \\ \text{through capillary} \\ \text{in viscometer} \end{array} \quad Q = \frac{\Delta V}{\Delta t} \quad (8.48)$$

The flow rate through a capillary as a function of system variables was solved for in the previous discussion, and the result is the Hagen-Poiseuille equation

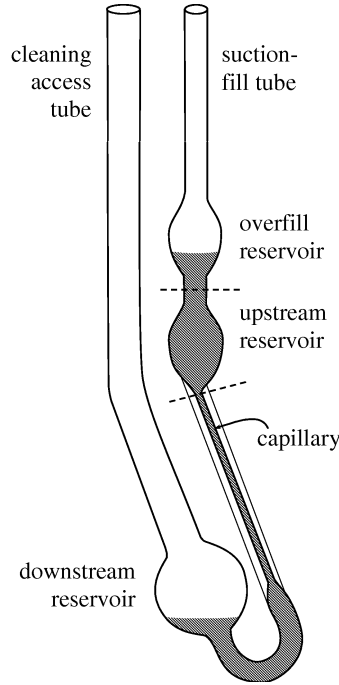


Figure 8.11: The Cannon-Fenske viscometer, a variation on the Ostwald viscometer invented by Wilhelm Ostwald, has two fluid reservoirs connected by a tilted capillary tube. The fluid is drawn up through the capillary to fill the second reservoir to overfull. The fluid level is then allowed to drop, and the time it takes for the fluid meniscus to pass between the two marks shown is the efflux time  $\Delta t$ .

(with gravity), equation 8.27.

$$\begin{array}{l} \text{Hagen-Poiseuille} \\ \text{(gravity } g_z = g \text{ included)} \end{array} \quad Q = \frac{\pi(p_0 - p_L + \rho g L)R^4}{8\mu L} \quad (8.49)$$

Because the capillary is tilted, gravity is not in the flow ( $\hat{e}_z$ ) direction, but is tilted from  $\hat{e}_z$  by an angle  $\theta$  (Figure 8.13). To apply equation 8.49 to the Cannon-Fenske system defined in Figure 8.12, we substitute the correct  $z$ -component of gravity ( $g \cos \theta$ ) for the  $z$ -component of gravity that was used in the derivation ( $g$ ).

$$\begin{array}{l} \text{Hagen-Poiseuille} \\ \text{(gravity } g_z = g \cos \theta \text{ included)} \end{array} \quad Q = \frac{\pi(p_0 - p_L + \rho g \cos \theta L)R^4}{8\mu L} \quad (8.50)$$

We now substitute  $Q$  from equation 8.48 and solve for  $\Delta t$ .

$$Q = \frac{\Delta V}{\Delta t} = \frac{\pi(p_0 - p_L + \rho g \cos \theta L)R^4}{8\mu L} \quad (8.51)$$

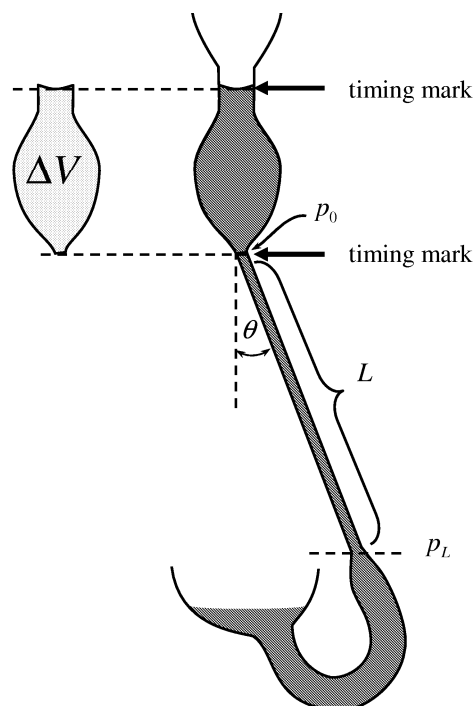


Figure 8.12: Schematic of the operation of a Cannon Fenske viscometer. A volume of fluid  $\Delta V$  is timed as it passes through a capillary of length  $L$ .

$$\Delta t = \frac{8\mu\Delta V}{\pi R^4} \left( \frac{1}{\frac{p_0 - p_L}{L} + \rho g \cos \theta} \right) \quad (8.52)$$

The pressure at the top of the capillary is very nearly atmospheric, and the pressure at the bottom of the capillary is also very nearly atmospheric; taking  $p_0 - p_L \approx 0$  (see problem 8.30 for corrections to this assumption), we obtain our final result.

$$t_{\text{efflux}} = \Delta t = \frac{8\mu\Delta V}{\pi R^4 \rho g \cos \theta} \quad (8.53)$$

Efflux time  
in Cannon-Fenske  
viscometer

$$\Delta t = \frac{8\nu\Delta V}{\pi R^4 g \cos \theta} \quad (8.54)$$

where  $\nu = \mu/\rho$  is the kinematic viscosity of the fluid. We can thus write the kinematic viscosity in terms of viscometer dimensions and the measured efflux

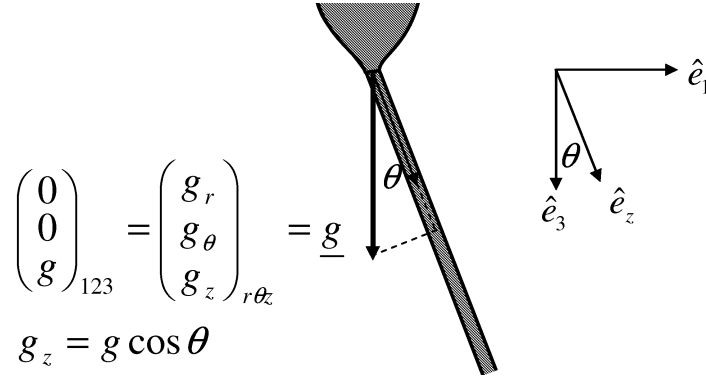


Figure 8.13: We can use geometry to relate the direction of the flow (along the tilted capillary, cylindrical coordinate  $\hat{e}_z$ ) to the direction of gravity (vertically down, Cartesian coordinate  $\hat{e}_3$ ).

time as follows:

Kinematic viscosity obtained with Cannon-Fenske viscometer ( $p_0 - p_L$ neglected)	$\nu = \left( \frac{\pi R^4 g \cos \theta}{8 \Delta V} \right) \Delta t$	(8.55)
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The angle  $\theta$  depends on how the viscometer is mounted during experimental operation; for that reason, experimentalists use great care in vertically aligning the viscometer. The values of  $R$  and  $\Delta V$  are fixed at the time of device manufacture.

Everything in parentheses in equation 8.55 is fixed for a given viscometer, although dimensions such as  $\Delta V$  and  $R$  vary slightly from instrument to instrument. As a matter of practicality, each viscometer is supplied by the manufacturer with a calibration constant that replaces the quantity in parentheses in equation 8.55. The calibration constant, which is a function of temperature, is determined at the factory by measuring efflux time  $\Delta t$  for a material of known kinematic viscosity  $\nu$ . This approach has the added advantage of accounting for the small neglected pressure difference, since the neglected pressure difference will have been present during calibration. Thus, the final operating equation for the Cannon-Fenske viscometer is

$$\nu = \left[ \left( \begin{array}{c} \text{correction} \\ \text{factor for} \\ p_0 - p_L \end{array} \right) \left( \frac{\pi R^4 g \cos \theta}{8 \Delta V} \right) \right] \Delta t \quad (8.56)$$

Kinematic viscosity obtained with Cannon-Fenske viscometer ( $p_0 - p_L$ accounted for)	$\nu(T) = \alpha(T)\Delta t$	(8.57)
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where  $\alpha(T)$  is the temperature-dependent calibration constant supplied by the manufacturer for a given viscometer. Note that in order for accurate viscosities to be measured, Cannon-Fenske viscometers must be charged with a standardized volume of material; excess fluid alters the back-pressure ( $p_L$ ) and introduces variability not accounted for by the calibration (see problem 8.31 for more discussion on this issue).

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The solution strategy of this section is general and may be used to solve for velocity and stress fields for well-defined flows. When the geometry or flow circumstances are complex, computer-implemented numerical methods[178] may be employed to solve the Navier-Stokes equations for  $\underline{v}$  and  $\underline{\tilde{\tau}}$ . We follow the methodology of this section to solve the Navier-Stokes equations for other flows in sections 8.2 and 8.3 and in Chapter 9.

We turn now to the burst-pipe problem and our need for information on turbulent flow.

### 8.1.2 Turbulent Flow in Pipes

In the previous section we sought to calculate the flow rate in the burst-pipe example using a result from laminar flow. The formula that we used was the Hagen-Poiseuille equation, an equation that relates pressure-drop to flow rate for laminar pipe flow.

Hagen-Poiseuille equation: ( $Q(\Delta p)$ for laminar tube-flow)	$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$	(8.58)
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We found that the burst-pipe result predicted by this laminar-flow equation was not correct; the predicted flow rate was almost two orders of magnitude too high. Our error in that calculation was to use a laminar-flow relationship to make a prediction in turbulent flow. To complete the burst-pipe calculation correctly, we need the flow-rate/pressure-drop relationship for turbulent flow in pipes.

We can seek the turbulent-flow flow-rate/pressure-drop relationship by following the same steps we used to develop the laminar-flow relationship. As we attempt to follow that process, we hope to see why and how the method fails in turbulent flow. The steps leading to the Hagen-Poiseuille equation (equation 8.58) were the following: