An Introduction to Fluid Mechanics

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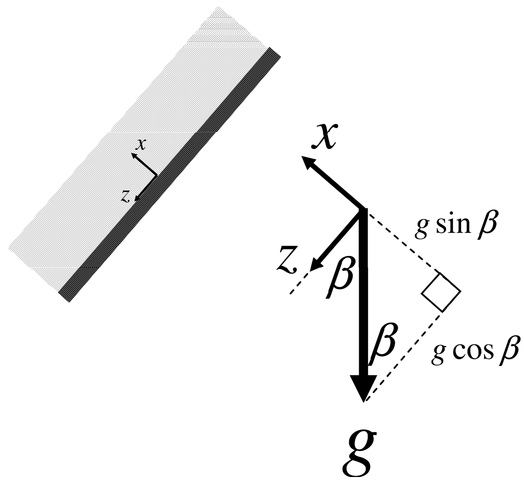

Figure 6.14: Because we have chosen a coordinate system that simplifies the velocity vector, the gravity vector is slightly more complicated than it might be with another choice of coordinate system.

To apply the free-surface boundary condition, we need to differentiate the result to obtain dv_z/dx.

$$\frac{dv_z}{dx} = \left[\frac{-\rho g \cos\beta}{\mu}\right] x + C_1 \tag{6.241}$$

If we choose $x = 0$ at the free surface, C_1 becomes zero, simplifying the algebra for determining the integration constants. If we choose $x = H$ at the free surface, we must perform more complex manipulations to obtain C_2.

Because of this advantage, it is customary to choose the origin for this problem at the free surface. When there is a symmetry plane or line of symmetry in a problem there will also be a boundary condition in terms of a derivative and the same logic applies.

6.2.3 Engineering Quantities from Velocity and Stress Fields

A final topic that may be of help to the student is a discussion of how to calculate engineering quantities from velocity and stress fields. Four important engineering quantities are: force on a surface, torque to produce a rotation, flow rate, and velocity/stress maxima.

6.2.3.1 Total Force on a Wall

One reason that fluids are used in devices is to transfer forces. An example of this that we have already discussed is the hydraulic lift (section 4.2.4.2), in which the fluid is used to amplify forces. Another example of fluids mediating forces is when a fluid is introduced between two solid parts as a lubricant to reduce the amount of force transferred (Figure 6.15). Sometimes force transfer is not the goal, for example when the transportation of the fluid

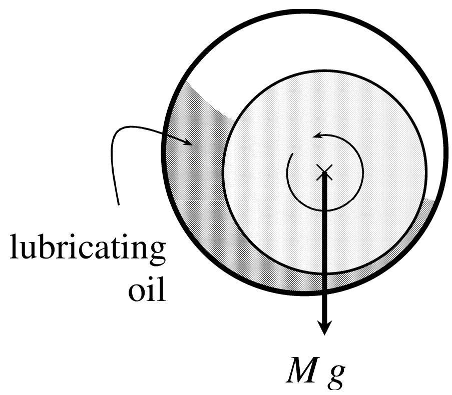

Figure 6.15: When two metal parts move relative to one another, such as in the journal bearing sketched above, a lubricant is used to reduce the stress transferred from one part to the other. The total force on a surface in contact with a lubricant can be calculated with the equations of this section.

itself is the engineering goal. In this case the forces of the fluid on the wall must be overcome with a pump or other device. In fast-moving equipment, the forces can be quite large and the consequences of failure disastrous.

In all of these examples, the design of the apparatus depends on knowing the total force that a fluid exerts on the wall. If the stress distribution in the fluid is obtained using the microscopic momentum balance, then the total force on any surface may be calculated by evaluating the fluid stress at each point on that surface and summing the product of stress and area over the entire surface.

We have dealt with such a sum over a surface before; the result is a surface integral. The force due to the fluid at one piece of the wall tangent plane ΔS_i is given by equation 4.265.

$$\begin{array}{c}\text{Fluid force}\\ \text{on surface } \Delta S_i\\ \text{with unit normal } \hat{n}\\ \text{at point } (x_i, y_i, z_i)\end{array} = \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{(x_i y_i z_i)} \Delta S_i \qquad (6.242)$$

where $\underline{\underline{\tilde{\Pi}}}$ is the total stress tensor, and $\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{x_i y_i z_i}$ is the stress on ΔS_i at x_i, y_i, z_i. To get the force on the entire wall, we sum all the pieces that make up the surface, and take the limit as $\Delta S = \Delta A/(\hat{n} \cdot \hat{e}_z)$ goes to zero (appendix C.2.1).

$$\begin{aligned}
\begin{array}{c}\text{Total fluid force}\\ \text{on a surface } \mathcal{S}\end{array} &= \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{(x_i y_i z_i)} \Delta S_i \right] && (6.243) \\
&= \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \frac{\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{(x_i y_i z_i)}}{\hat{n}_i \cdot \hat{e}_z} \Delta A_i \right] && (6.244) \\
&= \iint_{\mathcal{R}} \frac{\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}}}{\hat{n} \cdot \hat{e}_z} \, dA && (6.245) \\
&= \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS && (6.246)
\end{aligned}$$

$$\boxed{\begin{array}{c}\text{Total fluid force}\\ \text{on a surface } \mathcal{S}\end{array} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS} \qquad (6.247)$$

We previously introduced this expression in equation 4.288, and we use it extensively throughout the text.

We can try out equation 6.247 by calculating the total force on the incline in our falling-film example.

EXAMPLE 6.8 *What is the total vector force on the incline in the falling-film example (Figure 6.10)?*

SOLUTION The total force on a surface in a fluid is given by equation 6.247.

$$\begin{array}{c}\text{Total fluid force}\\ \text{on a surface } \mathcal{S}\end{array} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS \qquad (6.248)$$

The unit normal to the incline surface written in the chosen flow coordinate system (Figure 6.10) is $\hat{n} = \hat{e}_x$, and this unit normal vector is the same at every location on the surface of the incline. The stress tensor $\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$ was solved for in pieces in previous examples; the result for $\underline{\underline{\tilde{\Pi}}}$ can be constructed from equations 6.160 and 6.155. The final force then can be calculated with a straightforward integration of equation 6.248.

The total stress tensor $\underline{\underline{\tilde{\Pi}}}$ by definition is

$$\begin{aligned}
\underline{\underline{\tilde{\Pi}}} &= -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}} && (6.249) \\
&= \begin{pmatrix} -p(x) & 0 & \tilde{\tau}_{xz}(x) \\ 0 & -p(x) & 0 \\ \tilde{\tau}_{xz}(x) & 0 & -p(x) \end{pmatrix}_{xyz} && (6.250)
\end{aligned}$$

We solved previously for the two missing pieces of information, $p(x)$ and $\tilde{\tau}_{xz}$.

$$p(x) = p_{atm} + \rho g H \sin\beta \left(1 - \frac{x}{H}\right) \tag{6.251}$$

$$\tilde{\tau}_{xz}(x) = \rho g \cos\beta (H - x) \tag{6.252}$$

The surface of the incline is located at $x = 0$, and the unit normal to the entire surface is $\hat{n} = \hat{e}_x$. To use equation 6.248 we need $\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}$ at the incline surface.

$$\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{\text{at surface}} = \left[\hat{e}_x \cdot \underline{\underline{\tilde{\Pi}}}\right]_{x=0} \tag{6.253}$$

$$= \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{xyz} \cdot \begin{pmatrix} -p(0) & 0 & \tilde{\tau}_{xz}(0) \\ 0 & -p(0) & 0 \\ \tilde{\tau}_{xz}(0) & 0 & -p(0) \end{pmatrix}_{xyz}$$

$$= \begin{pmatrix} -p(0) & 0 & \tilde{\tau}_{xz}(0) \end{pmatrix}_{xyz} \tag{6.254}$$

$$= \begin{pmatrix} -p_{atm} - \rho g H \sin\beta \\ 0 \\ \rho g H \cos\beta \end{pmatrix}_{xyz} \tag{6.255}$$

Note that in the last step above we switched the vector from a row vector to a column vector for convenience.

The surface integral in equation 6.248 can be carried out by identifying dS for our surface and coordinate system. The surface of the incline is flat and rectangular, and therefore we write $dS = dydz$. We now complete the integration.

$$\begin{matrix}\text{Total force} \\ \text{on the} \\ \text{incline surface}\end{matrix} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{\text{at surface}} dS \tag{6.256}$$

$$= \int_0^L \int_0^W \begin{pmatrix} -p_{atm} - \rho g H \sin\beta \\ 0 \\ \rho g H \cos\beta \end{pmatrix}_{xyz} dydz \tag{6.257}$$

The limits of the integrals are chosen to cover the entire surface of the inclined plane. The quantities L and W are the length and the width of the incline. After integration the result is

$$\begin{matrix}\text{Total force} \\ \text{on the} \\ \text{incline surface}\end{matrix} = LW \begin{pmatrix} -p_{atm} - \rho g H \sin\beta \\ 0 \\ \rho g H \cos\beta \end{pmatrix}_{xyz} \tag{6.258}$$

$$= \left(-LW p_{atm} - LWH\rho g \sin\beta\right)\hat{e}_x + \rho g H \cos\beta \hat{e}_z \tag{6.259}$$

The integration was easy since nothing in the integral varies with y or z.

Looking back at the geometry of the falling-film problem and at the form of the solution in equation 6.258, we notice that we can rewrite our solution in a form that helps us to grasp its meaning.

$$\begin{aligned} \begin{array}{c}\text{Total fluid force}\\ \text{on the}\\ \text{incline surface}\end{array} &= \rho(LWH)\begin{pmatrix} -g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz} - LW\begin{pmatrix} p_{atm} \\ 0 \\ 0 \end{pmatrix}_{xyz} \\ &= \rho(LWH)\underline{g} - (LW)p_{atm}\hat{n} \end{aligned} \tag{6.260}$$

We see from this final way of writing the result that the force on the incline is just the weight of the fluid plus the force due to atmospheric pressure (Figure 6.16).

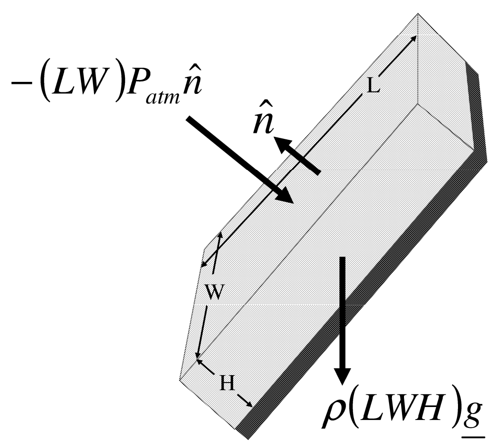

Figure 6.16: The force on the incline is a combination of the weight of the fluid and the force due to atmospheric pressure.

In the example 6.8 and with many calculations of this sort, we need to carry out a surface integration. The surface differential dS in equation 6.248 must be interpreted according to the specific case under consideration. As a convenience we assemble several common cases in Figure 6.17.

For the simple case of the flow down an incline, it appears that we could have arrived at the total force result by doing a straightforward force balance instead of performing the integration in equation 6.248. The surface-integration method is general, however, and is useful in more complex situations, including situations involving intricate wall shapes. We discuss a case involving spherical coordinates in example 6.9, and we discuss more flow examples in Chapters 7 and 8.

Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
Cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
Cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
Spherical, $\hat{n} = \hat{e}_r$	$dS = R^2 sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
Cylindrical	$dV = rdrd\theta dz$
Spherical	$dV = r^2 sin\theta drd\theta d\phi$

Figure 6.17: To carry out a surface or volume integration, the surface element dS or volume element dV must be written specifically for the coordinates in use, Cartesian, cylindrical, or spherical and for the surface under consideration.

EXAMPLE 6.9 *What is the total vector force on a sphere in creeping flow around a sphere, the flow shown in Figure 6.18?*

SOLUTION In Chapter 8 we discuss the solution for the velocity and stress fields for flow around a sphere. We can calculate the total force on the sphere from the results for $\underline{v}$ and $\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}}$ obtained there.

We begin with equation 6.247.

$$\begin{array}{c}\text{Total fluid force}\\ \text{in a fluid}\\ \text{on a surface } \mathcal{S}\end{array} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{\text{at surface}} dS \qquad (6.261)$$

As we stated above, we need $\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}}$ solved for with the microscopic momentum balance. If we presume that we have this result, then we can calculate the total force from equation 6.261. The surface in which we are interested is located at $r = R$ and has a outwardly pointing unit normal vector $\hat{n} = \hat{e}_r$, where we are using the spherical coordinate system as shown in Figure 6.18. The differential surface element dS on the surface of the sphere can be written in the spherical coordinate system as $dS = R^2 \sin\theta d\theta d\phi$ (Figure 6.17). The total force is then given by

$$\begin{array}{c}\text{Total fluid force}\\ \text{on the sphere}\end{array} = \int_0^{2\pi} \int_0^{\pi} \left[\hat{e}_r \cdot \underline{\underline{\tilde{\Pi}}}\right]_{r=R} R^2 \sin\theta d\theta d\phi \qquad (6.262)$$

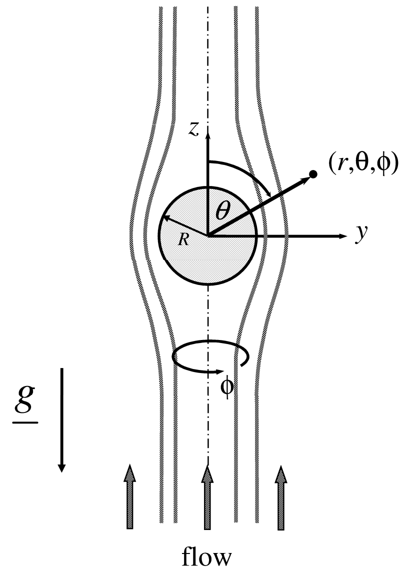

Figure 6.18: Flow around a sphere is important in droplet flow and in settling flows in suspensions.

The limits on the integrations are chosen to cover the entire surface of the sphere.

The velocity has nonzero components in the r and θ directions, but there is no swirling component in the ϕ-direction.

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_\phi \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} v_r \\ v_\theta \\ 0 \end{pmatrix}_{r\theta\phi} \tag{6.263}$$

The stress tensor in spherical coordinates is given in Table C.8, equation C.8-3. In this flow, there is symmetry in the ϕ-direction, which allows us to eliminate velocity derivatives with respect to ϕ in equation C.8-3. Also, $v_\phi = 0$; thus, four components of $\underline{\underline{\tilde{\tau}}}$ are zero. With these simplifications, equation C.8-3 becomes

$$\underline{\underline{\tilde{\tau}}} = \begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tilde{\tau}_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & 0 \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & 0 \\ 0 & 0 & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} \tag{6.264}$$

We can now write $\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} = \hat{e}_r \cdot (\underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}})$ as

$$\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} = \hat{e}_r \cdot \underline{\underline{\tilde{\Pi}}} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{r\theta\phi} \cdot \begin{pmatrix} \tilde{\tau}_{rr} - p & \tilde{\tau}_{r\theta} & 0 \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} - p & 0 \\ 0 & 0 & \tilde{\tau}_{\phi\phi - p} \end{pmatrix}_{r\theta\phi} \quad (6.265)$$

$$= \begin{pmatrix} \tilde{\tau}_{rr} - p & \tilde{\tau}_{r\theta} & 0 \end{pmatrix}_{r\theta\phi} \quad (6.266)$$

$$= \begin{pmatrix} \tilde{\tau}_{rr} - p \\ \tilde{\tau}_{r\theta} \\ 0 \end{pmatrix}_{r\theta\phi} \quad (6.267)$$

Substituting this result into equation 6.262, we obtain the expression that we must evaluate in order to get the force on the sphere.

$$\begin{array}{c}\text{Total fluid force} \\ \text{on the sphere}\end{array} = \int_0^{2\pi} \int_0^{\pi} \left[\hat{e}_r \cdot \underline{\underline{\tilde{\Pi}}} \right]_{r=R} R^2 \sin\theta d\theta d\phi \quad (6.268)$$

$$= \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} \tilde{\tau}_{rr}\big|_R - p\big|_R \\ \tilde{\tau}_{r\theta}\big|_R \\ 0 \end{pmatrix}_{r\theta\phi} R^2 \sin\theta d\theta d\phi \quad (6.269)$$

Without the microscopic-balance results for the components of $\underline{v}$, this is as far as we can go in our solution for the total force on the sphere. The solution to the flow around the sphere problem for creeping flow (slow flow) is given in Chapter 8. It turns out that $\tilde{\tau}_{rr}$ is equal to zero at the surface $r = R$, and thus the final expression to evaluate for force is that given below.

$$\boxed{\begin{array}{c}\text{Total fluid force} \\ \text{on the sphere}\end{array} = \int_0^{2\pi} \int_0^{\pi} \begin{pmatrix} -p\big|_R \\ \tilde{\tau}_{r\theta}\big|_R \\ 0 \end{pmatrix}_{r\theta\phi} R^2 \sin\theta d\theta d\phi} \quad (6.270)$$

We complete this calculation in Chapter 8.

Working on a more complex problem such as calculating the forces in the flow around a sphere is made considerably easier by having the general rule, equation 6.247, and then knowing how to apply it. There are more examples of the utility of equation 6.247 in the chapters ahead. In complex flows equation 6.247 is evaluated numerically with computer code[27, 173].

6.2.3.2 Torque

We worked briefly with torque in Chapter 1 (see example 1.17) and in Chapter 4 (example 4.20). As we saw in example 4.20, when parts turn in fluids, torque is needed or generated by the motion, and thus torque is an engineering quantity of interest in fluid mechanics.

Torque is the amount of effort to produce a rotation in a body; the definition of torque is the cross product of lever arm and the tangential force. The lever arm is the distance from the point of application of the force to the axis of rotation.

$$\underline{\mathcal{T}} = \text{(lever arm)} \times \text{(force)} \tag{6.271}$$

$$= \underline{R} \times \underline{f} \tag{6.272}$$

We calculate torque on a finite surface in a flow beginning with the fluid force on a infinitesimal surface given by equation 4.282.

$$\begin{array}{c}\text{Molecular fluid force}\\ \text{on surface } \Delta S_i\\ \text{with unit normal } \hat{n}\\ \text{at point } (x_i, y_i, z_i)\end{array} \qquad \underline{f}\big|_{\Delta S_i} = \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{(x_i y_i z_i)} \Delta S_i \tag{6.273}$$

The total torque is the sum of the infinitesimal torques on small pieces of the surface:

$$\begin{array}{c}\text{Total torque}\\ \text{on a surface } \mathcal{S}\end{array} = \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \underline{R}\big|_{\Delta S_i} \times \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{(x_i y_i z_i)} \Delta S_i \right] \tag{6.274}$$

$$= \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \frac{\underline{R}\big|_{\Delta A_i} \times \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{(x_i y_i z_i)}}{\hat{n}_i \cdot \hat{e}_z} \Delta A_i \right] \tag{6.275}$$

$$= \iint_{\mathcal{R}} \frac{\left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at surface}}}{\hat{n} \cdot \hat{e}_z} \, dA \tag{6.276}$$

$$= \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at surface}} dS \tag{6.277}$$

$$\boxed{\begin{array}{c}\text{Total torque}\\ \text{on a surface } \mathcal{S}\end{array} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at surface}} dS} \tag{6.278}$$

The torque may thus be calculated from the stress tensor, which, as usual, may be obtained from the solution of the momentum balance. We practice applying equation 6.278 in example 6.10.

EXAMPLE 6.10 *A compact device that may be used to measure viscosity and other flow properties of fluids is the parallel-plate rheometer (Figure 6.19). In this device, the gap between two circular disks is filled with fluid and one of* *the disks is turned. The design is such that the velocity field in the gap is given by*

$$\underline{v} = \begin{pmatrix} 0 \\ \frac{r\Omega z}{H} \\ 0 \end{pmatrix}_{r\theta z} = \frac{r\Omega z}{H}\hat{e}_\theta \tag{6.279}$$

The viscosity is related to the torque that is required to turn the disk. For a Newtonian fluid in such an apparatus, how is the viscosity related to the total torque to turn the top disk?

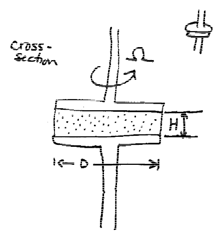

Figure 6.19: An incompressible, Newtonian fluid is confined between two circular disks of diameter D. The gap between the plates is H and the top plate rotates with a constant angular velocity Ω as shown in the figure.

SOLUTION The total torque on a surface in a fluid is given by equation 6.278:

$$\begin{matrix}\text{Total torque} \\ \text{on a surface } \mathcal{S}\end{matrix} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at surface}} dS \tag{6.280}$$

To apply this equation to calculate the torque in the current problem we identify each of the quantities in the equation and carry out the integration. Torque is needed to turn the top plate because the flat circular surface at $z = H$ is in contact with the fluid. The surface in the fluid in contact with the top plate has a unit normal $\hat{n} = \hat{e}_z$. The lever arm vector $\underline{R}$ is a vector from the axis of rotation to a point experiencing torque. The points on the surface experiencing torque are all the locations on the top fluid surface, and thus the lever arm is variable. We choose a small area $dS = rd\theta dr$, which is located a distance r from the axis of rotation. For this piece of area, the lever arm vector $\underline{R} = r\hat{e}_r$. Equation 6.280

becomes

$$\begin{array}{c}\text{Total torque on the}\\ \text{top fluid surface}\\ \text{in the parallel-plate}\\ \text{rheometer}\end{array} = \int_0^{2\pi}\int_0^{D/2}\left[\underline{R}\times\left(\hat{n}\cdot\underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at z=H}} dS \qquad (6.281)$$

$$= \int_0^{2\pi}\int_0^{D/2}\left[r\hat{e}_r\times\left(\hat{e}_z\cdot\underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at z=H}} r dr d\theta \qquad (6.282)$$

The stress tensor $\underline{\underline{\tilde{\Pi}}}$ comes from the Newtonian constitutive equation. Since we know that the velocity field (equation 6.280), we can calculate the expression we need directly from the constitutive equation, equation 5.103.

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}} \qquad (6.283)$$

$$= -p\underline{\underline{I}} + \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right) \qquad (6.284)$$

In cylindrical coordinates, the Newtonian constitutive equation is given in equation 5.105. We can immediately simplify equation 5.105 because $v_r = v_z = 0$ and v_θ is not a function of θ. With these simplifications, the Newtonian constitutive equation becomes

$$\tau = \begin{pmatrix} 0 & \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) & 0 \\ \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) & 0 & \mu\frac{\partial v_\theta}{\partial z} \\ 0 & \mu\frac{\partial v_\theta}{\partial z} & 0 \end{pmatrix}_{r\theta z} \qquad (6.285)$$

Carrying out the partial derivatives of $\underline{v}_\theta$ using the velocity field given and assembling $\underline{\underline{\tilde{\Pi}}}$ we obtain,

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right) \qquad (6.286)$$

$$= \begin{pmatrix} -p & \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) & 0 \\ \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) & -p & \mu\frac{\partial v_\theta}{\partial z} \\ 0 & \mu\frac{\partial v_\theta}{\partial z} & -p \end{pmatrix}_{r\theta z} \qquad (6.287)$$

$$\underline{\underline{\tilde{\Pi}}} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & \mu\frac{r\Omega}{H} \\ 0 & \mu\frac{r\Omega}{H} & -p \end{pmatrix}_{r\theta z} \qquad (6.288)$$

The next steps are to carry out the dot product and then the cross product in equation 6.282.

$$\begin{array}{c}\text{Total torque on the}\\ \text{top fluid surface}\\ \text{in the parallel-plate}\\ \text{rheometer}\end{array} = \int_0^{2\pi}\int_0^{D/2}\left[r\hat{e}_r \times \left(\hat{e}_z \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at z=H}} r dr d\theta \quad (6.289)$$

$$\hat{e}_z \cdot \underline{\underline{\tilde{\Pi}}} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}_{r\theta z} \cdot \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & \mu\frac{r\Omega}{H} \\ 0 & \mu\frac{r\Omega}{H} & -p \end{pmatrix}_{r\theta z} \quad (6.290)$$

$$= \begin{pmatrix} 0 & \mu\frac{r\Omega}{H} & -p \end{pmatrix}_{r\theta z} \quad (6.291)$$

$$r\hat{e}_r \times \left(\hat{e}_z \cdot \underline{\underline{\tilde{\Pi}}}\right) = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}_{r\theta z} \times \begin{pmatrix} 0 \\ \mu\frac{r\Omega}{H} \\ -p \end{pmatrix}_{r\theta z} \quad (6.292)$$

$$= \begin{pmatrix} 0 \\ rp \\ \frac{r^2\mu\Omega}{H} \end{pmatrix}_{r\theta z} \quad (6.293)$$

where we have used equation 1.188 to evaluate the cross product. The integral to evaluate thus becomes

$$\begin{array}{c}\text{Total torque on the}\\ \text{top fluid surface}\\ \text{in the parallel-plate}\\ \text{rheometer}\end{array} = \int_0^{2\pi}\int_0^{D/2}\left[r\hat{e}_r \times \left(\hat{e}_z \cdot \underline{\underline{\tilde{\Pi}}}\right)\right]_{\text{at z=H}} r dr d\theta \quad (6.294)$$

$$= \int_0^{2\pi}\int_0^{D/2}\begin{pmatrix} 0 \\ rp \\ \frac{r^2\mu\Omega}{H} \end{pmatrix}_{r\theta z} r dr d\theta \quad (6.295)$$

$$= \int_0^{2\pi}\int_0^{D/2}\left[r^2 p\hat{e}_\theta + \frac{r^3\mu\Omega}{H}\hat{e}_z\right] dr d\theta \quad (6.296)$$

The basis vector $\hat{e}_\theta$ is a function of θ; thus we convert to Cartesian coordinates before evaluating the integral.

$$\begin{array}{c}\text{Total torque on the}\\ \text{top fluid surface}\\ \text{in the parallel-plate}\\ \text{rheometer}\end{array} = \int_0^{2\pi}\int_0^{D/2}\left[r^2 p\hat{e}_\theta + \frac{r^3\mu\Omega}{H}\hat{e}_z\right] dr d\theta \quad (6.297)$$

$$= \int_0^{2\pi}\int_0^{D/2}\left[r^2 p\left(-\sin\theta\hat{e}_x + \cos\theta\hat{e}_y\right) + \frac{r^3\mu\Omega}{H}\hat{e}_z\right] dr d\theta \quad (6.298)$$

The θ-integral results in the $\hat{e}_x$ and $\hat{e}_y$ terms dropping out leaving the $\hat{e}_z$ component as the only nonzero component of torque. The details of the remaining steps are left to the reader. The final result for torque is

Total torque on the top fluid surface in the parallel-plate rheometer (Newtonian)

$$\underline{\mathcal{T}} = \frac{\pi\Omega\mu R^4}{2H}\hat{e}_z = \begin{pmatrix} 0 \\ 0 \\ \frac{\pi\Omega\mu R^4}{2H} \end{pmatrix}_{xyz} = \begin{pmatrix} 0 \\ 0 \\ \frac{\pi\Omega\mu R^4}{2H} \end{pmatrix}_{r\theta z} \quad (6.299)$$

We can therefore calculate the viscosity μ from a measurement of the torque as

Viscosity from torque in parallel-plate Newtonian

$$\mu = \frac{2H\mathcal{T}}{\pi\Omega R^4} \quad (6.300)$$

6.2.3.3 Flow Rate and Average Velocity

Forces and torques are two types of engineering variable; another important quantity is flow rate. Flow rate, or volume flow per unit time, may be calculated directly from a velocity profile. To calculate the flow rate through a finite surface $\mathcal{S}$ when the velocity varies across the surface, we once again calculate a surface integral.

The flow rate through one piece of the surface ΔS_i is given by equation 3.91 at that point.

Flow rate through surface ΔS_i with unit normal $\hat{n}$ at point (x_i, y_i, z_i)

$$= [\hat{n} \cdot \underline{v}]_{(x_i y_i z_i)} \;\; \Delta S_i \quad (6.301)$$

To get the total flow rate, we sum all the pieces that make up the surface $\mathcal{S}$, and take the limit as ΔS goes to zero (appendix C.2.1).

Total flow rate out through surface $\mathcal{S}$

$$\dot{V} = \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} [\hat{n} \cdot \underline{v}]_{(x_i y_i z_i)} \;\; \Delta S_i \right] \quad (6.302)$$

$$= \lim_{\Delta A \longrightarrow 0} \left[\sum_{i=1}^{N} \frac{[\hat{n} \cdot \underline{v}]_{(x_i y_i z_i)}}{\hat{n}_i \cdot \hat{e}_z} \Delta A_i \right] \quad (6.303)$$

$$= \iint_{\mathcal{R}} \frac{[\hat{n} \cdot \underline{v}]_{\text{at surface}}}{\hat{n} \cdot \hat{e}_z} \; dA \quad (6.304)$$

$$\dot{V} = \iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \;\; dS \quad (6.305)$$

Total flow rate out through surface $\mathcal{S}$	$\dot{V} = \iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \; dS$

(6.306)

To calculate the average velocity, we divide the total flow rate by the cross-sectional area of the flow.

Average velocity out through surface $\mathcal{S}$	$\langle v \rangle = \dfrac{\dot{V}}{\iint_{\mathcal{S}} dS} = \dfrac{\iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \; dS}{\iint_{\mathcal{S}} dS}$

(6.307)

We used this expression in equation **??** in the pipe-bend problem. We can try out these expressions by calculating the total flow rate and average velocity down the incline in our falling-film example.

EXAMPLE 6.11 *What are the flow rate and average velocity in the steady drag flow between parallel plates (see Example 6.4)?*

SOLUTION We begin with equation 6.306.

$$\begin{array}{c}\text{Total flow rate}\\ \text{out through}\\ \text{surface } \mathcal{S}\end{array} = \iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \; dS \tag{6.308}$$

We need to identify $\hat{n}$, $\underline{v}$, and the surface over which we wish to integrate. The flow rate in the drag-flow-between-infinite-plates problem is the same at every x_1-position throughout the flow, and therefore we can chose as our calculation surface any plane perpendicular to the flow; we choose the exit, $x_1 = L$. The unit normal of our calculation surface is $\hat{n} = \hat{e}_1$, and the velocity vector is given in equation 6.181 as $\underline{v} = (V/H)x_2\hat{e}_1$. The dot of these two vectors is

$$\hat{n} \cdot \underline{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{123} \cdot \begin{pmatrix} \frac{V}{H}x_2 \\ 0 \\ 0 \end{pmatrix}_{123} = \frac{V}{H}x_2 \tag{6.309}$$

The surface $\mathcal{S}$ is a rectangle in the 23-plane; thus $dS = dx_2 dx_3$ (Figure 6.17) and the location of the surface is $\underline{x}_1 = L$. The flow rate $\dot{V}$ is then given by

$$\dot{V} = \iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{x_1 = L} \; dS \tag{6.310}$$

$$= \int_0^W \int_0^H \left(\frac{V}{H} x_2 \right) dx_2 dx_3 \tag{6.311}$$

$$= \frac{WHV}{2} \tag{6.312}$$

The details of the integration are left to the reader (see problem 6.7). The average velocity is just $\dot{V}/HW$.

Average velocity out through surface $\mathcal{S}$ in drag flow

$$\langle v \rangle = \frac{\iint_{\mathcal{S}} [\hat{n} \cdot \underline{v}]_{\text{at surface}} \; dS}{\iint_{\mathcal{S}} dS} \tag{6.313}$$

$$= \frac{\dot{V}}{\int_0^W \int_0^H dx_1 dx_2} \tag{6.314}$$

$$= \frac{V}{2} \tag{6.315}$$

6.2.3.4 Velocity and Stress Extrema

In some engineering problems, the maximum or minimum velocity or force is of interest. For example, if a fluid jet hits a surface, the maximum value of the force would be important to know in designing the surface to withstand the impact. The location of the maximum force is also important when designing a bracing system for such a device.

To locate the maximum or minimum of any function (for example velocity or stress component), we calculate the first derivative of the function and set it equal to zero (section 1.3.1)[148].

$$\text{At the maximum/minimum of } f(x): \qquad \frac{df}{dx} = 0 \tag{6.316}$$

Solving equation 6.316 for $x_{min/max}$ gives us the location of the minimum or maximum. To determine if the extrema located is a minimum or a maximum, we calculate the second derivative[148].

$$\left.\frac{d^2 f}{dx^2}\right|_{x_{min/max}} > 0 \quad \Rightarrow \text{minimum} \tag{6.317}$$

$$\left.\frac{d^2 f}{dx^2}\right|_{x_{min/max}} < 0 \quad \Rightarrow \text{maximum} \tag{6.318}$$

EXAMPLE 6.12 *A Newtonian fluid flows steadily between two long, wide plates under an imposed pressure difference* $\Delta p = p_0 - p_L$ *(Figure 6.20). In addition, the top plate moves at a velocity* V*. The velocity field may be found* *by using the methods of Chapter 7, and the solution for $v_x(y)$ in the coordinate system of Figure 6.20 is given below.*

$$v_x(y) = \frac{H^2(p_L - p_0)}{2\mu L}\left[\left(\frac{y}{H}\right)^2 - 1\right] + \frac{V}{2}\left(\frac{y}{H} + 1\right) \quad (6.319)$$

What is the location of the velocity maximum as a function of the imposed pressure difference?

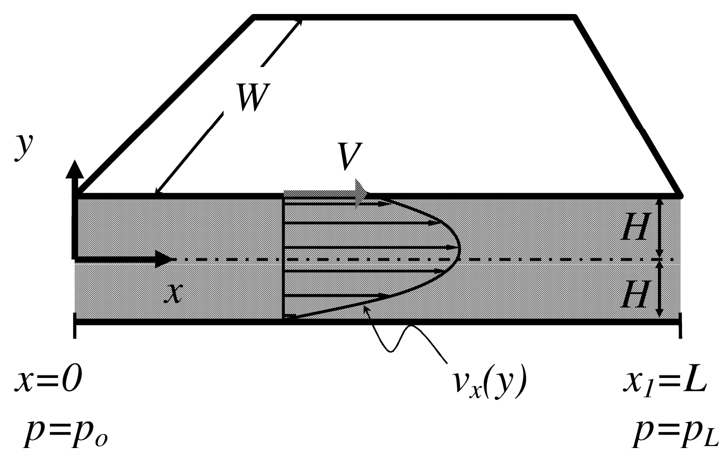

Figure 6.20: Combined pressure and drag flow of a Newtonian fluid through a wide, long slit may be modeled as shown.

SOLUTION To find the location y_{max} at which the velocity function $v_x(y)$ attains its maximum value, we need to find the location where the first derivative of v_x with respect to y goes to zero.

$$v_x(y) = \frac{H^2(p_L - p_0)}{2\mu L}\left[\left(\frac{y}{H}\right)^2 - 1\right] + \frac{V}{2}\left(\frac{y}{H} + 1\right) \quad (6.320)$$

location of maximum:
$$\left.\frac{dv_x}{dy}\right|_{y=y_{max}} = 0 \quad (6.321)$$

To simplify the algebra, we define the constant B as

$$B \equiv \frac{H^2(p_L - p_0)}{2\mu L} \quad (6.322)$$

which allows us to write the velocity as

$$v_x(y) = B\frac{y^2}{H^2} - B + \frac{V}{2H}y + \frac{V}{2} \quad (6.323)$$

We now take the first derivative of $v_x(y)$ and solve for the value of y that makes this zero.

$$\frac{dv_x}{dy} = \frac{2B}{H^2}y + \frac{V}{2H} \tag{6.324}$$

$$0 = \frac{2B}{H^2}y_{max} + \frac{V}{2H} \tag{6.325}$$

$$y_{max} = \frac{-VH}{4B} \tag{6.326}$$

$$\boxed{y_{max} = \frac{\mu L V}{2H(p_0 - p_L)}} \tag{6.327}$$

We can verify that this is in fact a maximum rather than a minimum by calculating the second derivative of $v_x(y)$.

$$\frac{d^2 v_x}{dy^2} = \frac{2B}{H^2} \tag{6.328}$$

$$\frac{d^2 v_x}{dy^2} = \frac{p_L - p_0}{\mu L} < 0 \tag{6.329}$$

Since the upstream pressure is higher than the downstream pressure ($p_0 > p_L$), equation 6.329 tells us that the second derivative is negative throughout the flow; thus, the extremum we have found is a maximum (compare to equation 6.318).

6.3 Summary

In this chapter we have derived and used the microscopic mass and momentum balances. For fluids in general, the microscopic momentum balance is the Cauchy momentum equation, equation 6.127. For incompressible, Newtonian fluids, the microscopic momentum balance is the Navier-Stokes equation, equation 6.169. We have shown how to apply these equations to a problem with which we are familiar, the flow of a thin film down an inclined plane. We have also discussed two topics that are needed to effectively apply the microscopic balances: flow boundary conditions and methods for calculating macroscopic engineering properties from the microscopic results.

We have laid the groundwork for performing microscopic balances on a wide variety of flows. In the next two chapters we discuss microscopic solutions in three important flow classes, internal flows, external flows, and boundary-layer flows. In those chapters we apply the microscopic balances to simple cases, we apply the microscopic balances to more complicated cases, and we discuss how to use dimensional analysis to modify the microscopic analysis when a detailed microscopic solution is impractical or unnecessary.