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EXAMPLE: What is the inverse-deformation gradient tensor in steady shear flow?

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 y \\ 0 \\ 0 \end{pmatrix}_{xyz} \qquad \underline{r} = \begin{pmatrix} x' + (t - t')\dot{\gamma}_0 y' \\ y' \\ z' \end{pmatrix}_{xyz}$$

Calculate $\underline{\underline{F}}$, $\underline{\underline{F}}^{-1}$ for shear

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz} = \begin{pmatrix} x' + \underbrace{(t-t')}_{\text{time dif.}} \underbrace{\dot{\gamma}_0}_{\text{velocity}} y' \\ y' \\ z' \end{pmatrix}_{xyz}$$

$$\underline{\underline{F}} = \frac{\partial \underline{r}'}{\partial \underline{r}} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz}$$

Need to invert this

$$x' + (t-t')\dot{\gamma}_0 y' = x$$
$$y' = y$$
$$z' = z$$

SOLVE FOR x', y', z'

Solve for x', y', z' as a function of x, y, z

$$\boxed{\begin{aligned} x' &= x - (t-t')\dot{\gamma}_0\, y \\ y' &= y \\ z' &= z \end{aligned}}$$

Now, carry out derivatives in $\underline{\underline{F}}$

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -(t-t')\dot{\gamma}_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

Let $\gamma = \int_{t'}^{t} \dot{\gamma}_0 \, dt'' = \dot{\gamma}_0 (t-t')$

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

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tensor	shear in 1-direction with gradient in 2-direction	uniaxial elongation in 3-direction	ccw rotation around $\hat{e}_3$
$\underline{\underline{F}}(t,t')$	$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{-\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{F}}^{-1}(t',t)$	$\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{C}}(t,t')$	$\begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon} & 0 & 0 \\ 0 & e^{\epsilon} & 0 \\ 0 & 0 & e^{-2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{C}}^{-1}(t',t)$	$\begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon} & 0 & 0 \\ 0 & e^{-\epsilon} & 0 \\ 0 & 0 & e^{2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{\gamma}}^{[o]}(t,t')$	$\begin{pmatrix} 0 & -\gamma & 0 \\ -\gamma & \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon}-1 & 0 & 0 \\ 0 & e^{\epsilon}-1 & 0 \\ 0 & 0 & e^{-2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$
$\underline{\underline{\gamma}}_{[o]}(t,t')$	$\begin{pmatrix} -\gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon}-1 & 0 & 0 \\ 0 & e^{-\epsilon}-1 & 0 \\ 0 & 0 & e^{2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$

Table 9.3: Strain tensors for shear and extension in Cartesian coordinates. For shear flows $\gamma = \gamma(t',t) = \int_{t'}^{t} \dot{\varsigma}(t'')\, dt'' = \int_{t'}^{t} \dot{\gamma}_{21}(t'')\, dt''$ and for elongational flows $\epsilon = \epsilon(t',t) = \int_{t'}^{t} \dot{\epsilon}(t'')\, dt''$. The angle ψ is the angle from $\underline{r}(t) = \underline{r}$ to $\underline{r}(t') = \underline{r}'$ in counter-clockwise (ccw) rotation around the $\hat{e}_3$-axis.

Swap