Show that:

$$d\underline{r}' \cdot \underline{\underline{F}}^{-1} = d\underline{r}.$$

inverse deformation gradient tensor

② ①

$\Rightarrow$ consider the position of a particle at time t'

To identify which particle I'm talking about, I'll use its position at t

$\underline{r}'(t'; \underline{r})$ = position at t' of the particle that at t was at position $\underline{r}$

$$\underline{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\tilde{x}\tilde{y}\tilde{z}} \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz}$$

$$d\underline{r}' = \begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix}_{xyz}$$

write using chain rule

$$\underline{r}' = \underline{r}'(t', \underline{r})$$

$$dx' = \frac{\partial x'}{\partial x}\,dx + \frac{\partial x'}{\partial y}\,dy + \frac{\partial x'}{\partial z}\,dz$$

$$dy' = \frac{\partial y'}{\partial x}\,dx + \frac{\partial y'}{\partial y}\,dy + \frac{\partial y'}{\partial z}\,dz$$

$$dz' = \frac{\partial z'}{\partial x}\,dx + \frac{\partial z'}{\partial y}\,dy + \frac{\partial z'}{\partial z}\,dz$$

NOTE – this can be written as,

$$dx' = d\underline{r}\cdot\frac{\partial x'}{\partial \underline{r}}\ \cancel{d\underline{r}}$$

$$dx_p\,\hat{e}_p\ \frac{\partial x'}{\partial r_i}\,\hat{e}_i$$

$$= \cancel{\frac{\partial x'}{\partial x_i}\,\underbrace{\hat{e}_i\cdot d x_p\,\hat{e}_p}_{\delta_{ip}}} = \frac{\partial x'}{\partial x_p}\,dx_p$$

$$dy' = d\underline{r}\cdot\frac{\partial y'}{\partial \underline{r}}\ \cancel{d\underline{r}}$$

$$dz' = d\underline{r}\cdot\frac{\partial z'}{\partial \underline{r}}\ \cancel{d\underline{r}}$$

$\equiv \underline{\underline{F}}$

OR $$\boxed{d\underline{r}' = d\underline{r}\cdot\frac{\partial \underline{r}'}{\partial \underline{r}}\ \cancel{d\underline{r}}}$$

$$\boxed{d\underline{r}' = d\underline{r}\cdot\underline{\underline{F}}\ \cancel{d\underline{r}}}$$

Let $\underline{\underline{F}}^{-1}$ be the inverse of $\underline{\underline{F}}$ (4)

$$d\underline{r}' = d\underline{r}\cdot\underline{\underline{F}} \quad \cancel{\cdot d\underline{r}}$$

$$d\underline{r}'\cdot\underline{\underline{F}}^{-1} = d\underline{r}\cdot\underline{\underline{F}}\cdot\underline{\underline{F}}^{-1}$$

$$\cancel{\underline{\underline{F}}^{-1}\cdot d\underline{r}' = \underbrace{\underline{\underline{F}}^{-1}\cdot\underline{\underline{F}}}_{\underline{\underline{I}}}\cdot d\underline{r}}$$

$$\underline{\underline{F}}^{-1}\cdot\underline{\underline{F}} = \underline{\underline{I}}$$

$$\underline{\underline{F}}\cdot\underline{\underline{F}}^{-1} = \underline{\underline{I}}$$

$$\boxed{d\underline{r}'\cdot\underline{\underline{F}}^{-1} \cancel{\cdot d\underline{r}'} = d\underline{r}}$$

inverse deformation gradient tensor

compare:

$$d\underline{r}\cdot\underline{\underline{F}} \cancel{\cdot d\underline{r}} = d\underline{r}'$$