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EXAMPLE: Calculate the material functions of steady shear flow for the Lodge model.

Lodge Model: $\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$

You try.

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Strain-centric "recipe card"

Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \gamma_0 = \text{constant}$$

$$\underline{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{123} \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{123} = \begin{pmatrix} x' + \dot{\gamma}_0 (t - t') \\ y' \\ z' \end{pmatrix}_{123}$$

$$\underline{\underline{F}}^{-1}(t', t) = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123} \qquad \underline{\underline{C}}^{-1}(t', t) = \begin{pmatrix} 1 + \gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123} \qquad \gamma = \dot{\gamma}_0 (t - t')$$

Material Functions:

Viscosity	First normal-stress coefficient	Second normal-stress coefficient
$\eta \equiv \dfrac{-\tau_{21}}{\dot{\gamma}_0}$	$\Psi_1 \equiv \dfrac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$	$\Psi_2 \equiv \dfrac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$

What is γ? (see caption, Table 9.3)

$$\gamma = \int_{t'}^{t} \dot{\gamma}(t'')\,dt'' = \int_{t'}^{t} \dot{\gamma}_0 \, dt''$$

$$= \dot{\gamma}_0 \, t'' \Big|_{t'}^{t}$$

$$\boxed{\gamma = \dot{\gamma}_0 \, (t - t')}$$

$$-\underline{\underline{\tau}}(t)\,\frac{\lambda^2}{\eta_0} = \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} \overbrace{\begin{pmatrix} 1 + \dot{\gamma}_0^2 (t-t')^2 & \dot{\gamma}_0 (t-t') & 0 \\ \dot{\gamma}_0 (t-t') & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}}^{\underline{\underline{C}}^{-1}(t',t)} dt'$$

$$\tau_{22} = \tau_{33} \Rightarrow \tau_{22} - \tau_{33} = 0 \Rightarrow \boxed{\Psi_2 = 0}$$

$$-\tau_{21} \frac{\lambda^2}{\eta_0} = \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} \dot{\gamma}_0 \, (t-t') \, dt'$$

$$-\tau_{21} \frac{\lambda^2}{\eta_0 \dot{\gamma}_0} = \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} (t-t') \, dt'$$

let $s = t - t'$

$ds = -dt'$

$t' = -\infty \, , \; s = \infty$

$t' = t \, , \; s = 0$

$$-\frac{\tau_{21}}{\dot{\gamma}_0}\frac{\lambda^2}{\eta_0} = -\int_{\infty}^{0} e^{-s/\lambda} s \, ds$$

$$= \lambda^2 \int_0^{\infty} \underbrace{\left(\frac{-1}{\lambda}\right) s \, e^{-s/\lambda} \left(\frac{-1}{\lambda}\right) ds}_{\text{from Wolfram Alfa}}$$

$$\int x e^x \, dx = e^x (x-1) + c$$

$$x = \frac{-s}{\lambda} \quad dx = \frac{-1}{\lambda} ds$$

$$= \lambda^2 e^{-s/\lambda} \left(\frac{-s}{\lambda} - 1\right)\Big|_0^{\infty}$$

$$= \lambda^2 \Big(\underbrace{0 - (-1)(1)}_{1} \Big) = \lambda^2$$

$$-\frac{\tau_{21}}{\dot{\gamma}_0} = \eta = \cancel{\lambda^2}\left(\frac{\eta_0}{\cancel{\lambda^2}}\right)$$

$$\boxed{\eta = \eta_0}$$

$$\tau_{11} - \tau_{22} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \left[\cancel{1} + \dot{\gamma}_0^2 (t-t')^2 - \cancel{1}\right] dt'$$

$$-\frac{(\tau_{11} - \tau_{22})\lambda^2}{\eta_0 \dot{\gamma}_0^2} = \int_{-\infty}^{t} e^{-\frac{(t-t')}{\lambda}} (t-t')^2 \, dt'$$

let $s = t - t'$

$ds = -dt'$

$t' = -\infty, \; s = \infty$

$t' = t, \; s = 0$

$$\frac{-(\tau_{11}-\tau_{22})\lambda^2}{\eta_0 \dot{\gamma}_0^2} = -\int_{\infty}^{0} e^{-s/\lambda} s^2 \, ds$$

$$= -\lambda^3 \int_0^\infty \underbrace{\frac{1}{\lambda^2} s^2 e^{-s/\lambda} \left(-\frac{1}{\lambda}\right) ds}_{\text{Wolfram Alfa}}$$

$$\int x^2 e^x \, dx = e^x(x^2-2x+2)+c$$

$x = -\frac{s}{\lambda}$ $\quad dx = -\frac{1}{\lambda} d_x$

$x^2 = \frac{s^2}{\lambda^2}$

$$= -\lambda^3 \left[e^{-s/\lambda}(s^2-2s+2) \right] \Big|_0^\infty$$

$$\frac{-(\tau_{11}-\tau_{22})\cancel{\lambda^2}}{\eta_0\dot{\gamma}_0^{\,2}} = -\lambda^3\left[0-\{2\}\right]$$

$$= 2\lambda^{\cancel{3}}$$

$$\frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0} = \boxed{\Psi_1 = 2\lambda\eta_0}$$

<u>Summary</u> <u>Lodge model</u> in steady shear.

$$\boxed{\begin{aligned}\eta &= \eta_0\\ \Psi_1 &= 2\lambda\eta_0\\ \Psi_2 &= 0\end{aligned}}$$

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