We have made considerable progress in our quest to relate red-fluid momentum changes to momentum changes of the fluid in the control volume. To proceed further, we write mathematical expressions for the two quantities expressed in words on the right-hand side of equation 3.67. These two quantities are entering and exiting fluid momenta at $t + \Delta t$, that is, momenta of fluid that crosses the control-volume boundaries. Both of these expressions can be written following the same approach; the calculation results in a double integral over the control-volume bounding surfaces.

The final mathematical expression for the terms in equation 3.67 are given in equation 3.133, and some readers on first reading may choose to proceed ahead at this point to section 3.2.3.[5] We derive these expressions in the discussion below.

3.2.2.2 The Convective Term

To convert the word expressions in equation 3.67 to mathematical terms, we need to consider how to use the continuum model to keep track of mass or momentum flow in through a surface. We begin by considering the simplest case of direct mass and momentum flow through a flat surface.

EXAMPLE 3.6 *Liquid passes through a chosen area A as shown in Figure 3.24. The velocity is perpendicular to the surface A at every point and does not vary across the cross-section. What are the volumetric flow rate (volume liquid/time), mass flow rate (mass/time), and momentum flow rate (momentum/time) through A?*

SOLUTION Figure 3.24 shows that for the case under consideration, the velocity of the fluid is perpendicular to the surface A and is constant (does not vary with position). Consider the fluid that passes through A during a short time interval Δt (Figure 3.25). The volume of fluid that passes through A during the interval Δt forms a solid whose volume is given by

$$\begin{pmatrix} \text{Volume of fluid} \\ \text{passing through } A \\ \text{in time } \Delta t \end{pmatrix} = \begin{pmatrix} \text{height} \\ \text{of solid} \end{pmatrix} \begin{pmatrix} \text{cross-section} \\ \text{of solid} \end{pmatrix} \tag{3.68}$$

$$= \Delta x \, A \tag{3.69}$$

where Δx is the change in location of fluid that started at A and has moved in the x-direction for time Δt. The magnitude of the fluid velocity, v, can be

[5] The two examples we are about to discuss deal with some very basic relationships of fluid mechanics and are worth spending some time on, even if you choose to skip the subsequent derivation.

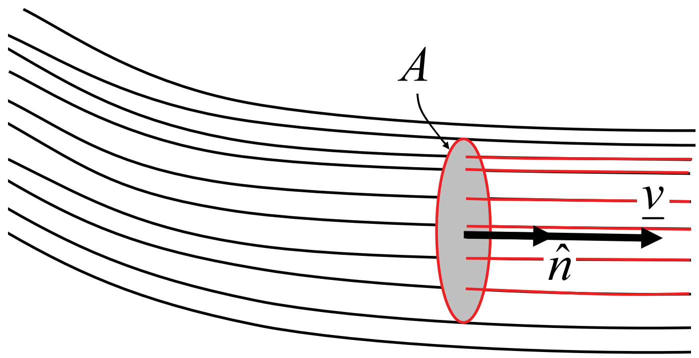

Figure 3.24: For this example we consider the flow through a surface A. The velocity of the fluid is perpendicular to the surface A.

written as

$$\text{Magnitude of fluid velocity} \qquad |\underline{v}| = v = \frac{\Delta x}{\Delta t} \tag{3.70}$$

With these two expressions we can calculate all the quantities of interest. The volumetric flow rate is the volume of fluid divided by the time interval.

$$Q = \frac{\text{fluid volume}}{\text{time interval}} = \frac{\Delta x\ A}{\Delta t} = v\ A \tag{3.71}$$

$$\begin{array}{c}\text{Volumetric flow}\\ \text{of liquid through } A\\ \text{(velocity perpendicular to } A\text{;}\\ \underline{v} \text{ does not vary across } A\text{)}\end{array} \qquad \boxed{Q = v\ A} \tag{3.72}$$

The mass flow rate can be calculated from the volumetric flow rate and the density.

$$m = \left(\frac{\text{mass}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.73}$$

$$= (\rho)(v\ A) \tag{3.74}$$

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\\ \text{(velocity perpendicular to } A\text{;}\\ \underline{v} \text{ does not vary across } A\text{)}\end{array} \qquad \boxed{m = (\rho)(v\ A)} \tag{3.75}$$

Finally, the momentum flow rate (a vector quantity) can be calculated from the

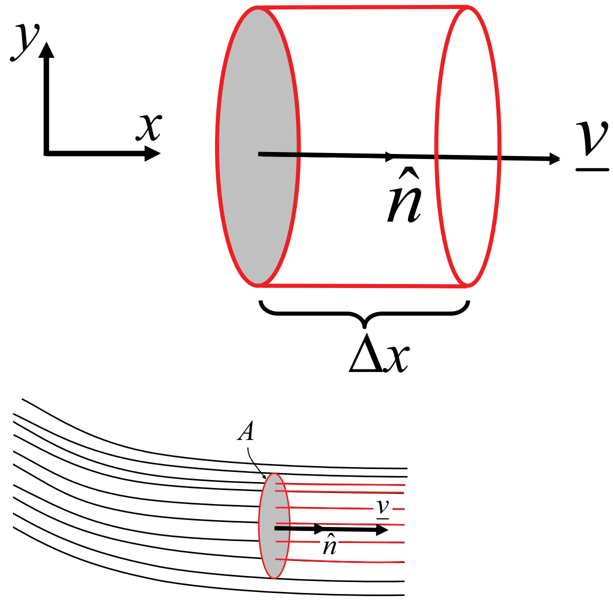

Figure 3.25: During the time interval Δt, a volume of fluid of height Δx and of cross-sectional area A passes through the area A.

definition of momentum and the previous results.

$$\begin{aligned}
\begin{pmatrix} \text{Momentum flow} \\ \text{of liquid through } A \end{pmatrix} &= \left(\frac{\text{momentum}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) && (3.76)\\
&= \frac{(\text{mass})(\text{velocity})}{\text{volume}}\left(\frac{\text{volume}}{\text{time}}\right) && (3.77)\\
&= \left(\frac{\text{mass}}{\text{volume}}\right)(\underline{v})\left(\frac{\text{volume}}{\text{time}}\right) && (3.78)\\
&= \rho\, \underline{v}\, (vA) && (3.79)
\end{aligned}$$

Note that for this example the velocity of the fluid was perpendicular to the surface A and $\underline{v}$ does not vary across A.

$$\boxed{\begin{matrix} \text{Momentum flow} \\ \text{of liquid through } A \\ (\text{velocity perpendicular to } A; \\ \underline{v} \text{ does not vary across } A) \end{matrix} = \rho\, \underline{v}\, (vA)} \qquad (3.80)$$

The previous example shows how powerful the continuum approach is. With very simple logic (essentially unit matching), we are able to express volume, mass, and momentum flows for a chosen system in terms of two field variables, density and velocity. For more complex systems, we build on these relationships and employ some vector tools, as we show in the next example.

EXAMPLE 3.7 *Liquid passes through a chosen area A as shown in Figure 3.26. The velocity of the fluid makes an angle θ with the unit normal to A, which is called $\hat{n}$. The velocity does not vary across the surface A. What are the volumetric flow rate (volume liquid/time), mass flow rate (mass/time), and momentum flow rate (momentum/time) through A?*

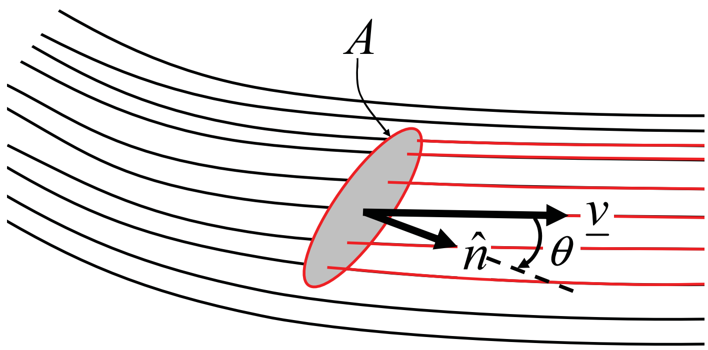

Figure 3.26: For this example we consider the flow through a surface A. The velocity of the fluid is not perpendicular to the surface A; instead, the velocity makes an angle θ with the surface unit normal $\hat{n}$.

SOLUTION The logic of the solution is the same for this case as in the previous example; there is, however, a difference in the volume of fluid that passes through A in time interval Δt.

Consider the fluid that passes through A during the short time interval Δt (Figure 3.27). The x-direction is the direction of flow. In time interval Δt fluid that started on the surface A moved along x a distance $\cos\theta\Delta x$. The volume of fluid that passed through A in this time interval is the volume of the solid shown. The volume of fluid that passes through A during the interval Δt is thus given by

$$\begin{pmatrix} \text{Volume of fluid} \\ \text{passing through } A \\ \text{in time } \Delta t \end{pmatrix} = \begin{pmatrix} \text{height} \\ \text{of solid} \end{pmatrix} \begin{pmatrix} \text{cross-section} \\ \text{of solid} \end{pmatrix} \tag{3.81}$$

$$= (\Delta x \cos\theta)\ A \tag{3.82}$$

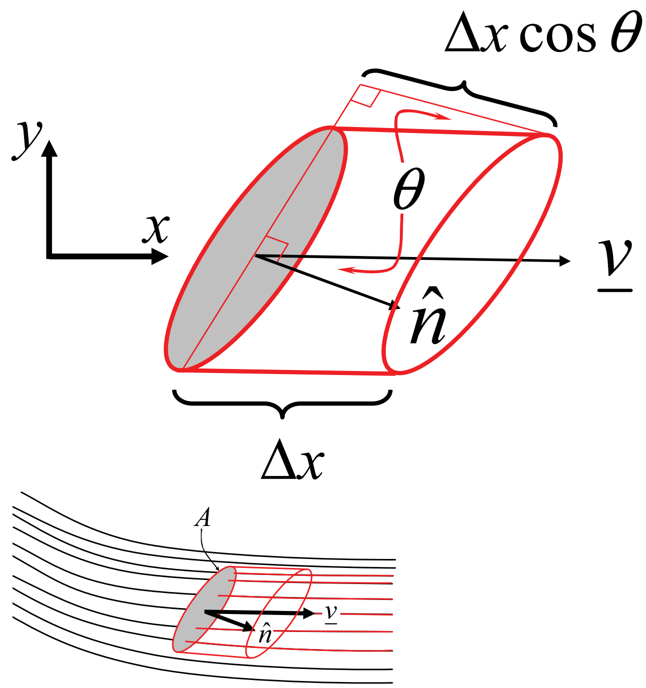

Figure 3.27: During the time interval Δt, a volume of fluid of height $\Delta x \cos\theta$ and of cross-sectional area A passes through the area A.

The magnitude of the fluid velocity, v, can be written as before as

$$\text{Magnitude of fluid velocity} \qquad |\underline{v}| = v = \frac{\Delta x}{\Delta t} \tag{3.83}$$

With these two expressions we can calculate all the quantities of interest.

$$\begin{aligned} \text{Volumetric flow of liquid through } A \qquad Q &= \frac{\text{fluid volume}}{\text{time interval}} && (3.84) \\ &= \frac{\Delta x \cos\theta \ A}{\Delta t} && (3.85) \\ &= v\cos\theta \ A && (3.86) \\ &= (\hat{n}\cdot\underline{v})A && (3.87) \end{aligned}$$

$$\begin{array}{c}\text{Volumetric flow} \\ \text{of liquid through } A \\ \text{(general orientation case;} \\ \underline{v} \text{ does not vary across } A\text{)}\end{array} \qquad \boxed{Q = v\cos\theta \ A = (\hat{n}\cdot\underline{v})A} \tag{3.88}$$

We have used the definition of the dot product to write the final result (equation 3.86) in vector notation ($\hat{n}\cdot\underline{v} = |\hat{n}||\underline{v}|\cos\theta = v\cos\theta$; see equation 1.166). As before, the mass flow rate can be calculated from the volumetric flow rate and the density.

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\end{array} \qquad m = \left(\frac{\text{mass}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.89}$$

$$= (\rho)\ (v\cos\theta\ A) = \rho\ (\hat{n}\cdot\underline{v})\ A \tag{3.90}$$

$$\begin{array}{c}\text{Mass flow}\\ \text{of liquid through } A\\ \text{(general orientation case;}\\ \underline{v} \text{ does not vary across } A)\end{array} \qquad \boxed{m = \rho\ (\hat{n}\cdot\underline{v})\ A} \tag{3.91}$$

Finally, the momentum flow rate can be calculated as before from the definition of momentum and the previous results.

$$\left(\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\end{array}\right) = \left(\frac{\text{momentum}}{\text{volume}}\right)\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.92}$$

$$= \frac{(\text{mass})(\text{velocity})}{\text{volume}}\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.93}$$

$$= \left(\frac{\text{mass}}{\text{volume}}\right)(\underline{v})\left(\frac{\text{volume}}{\text{time}}\right) \tag{3.94}$$

$$= \rho\ \underline{v}\ (v\cos\theta\ A) = \rho\ \underline{v}\ (\hat{n}\cdot\underline{v})A \tag{3.95}$$

This is the general result when $\underline{v}$ is not necessarily perpendicular to A.

$$\boxed{\left(\begin{array}{c}\text{Momentum flow}\\ \text{of liquid through } A\\ \text{(general orientation case;}\\ \underline{v} \text{ does not vary across } A)\end{array}\right) = \rho\underline{v}\ (\hat{n}\cdot\underline{v})A} \tag{3.96}$$

We recover the case of velocity perpendicular to A (equation 3.80) when $\theta = 0$ ($\cos 0 = 1$, $\hat{n}\cdot\underline{v} = v$).

The relationship we obtained in equation 3.88 for volumetric flow rate through an area as a function of the locally constant velocity $\underline{v}$ ($Q = (\hat{n}\cdot\underline{v})A$) is similar to an equation introduced in Chapter 1 that relates overall volumetric flow rate through a pipe to the *average* velocity in the pipe $\langle v\rangle$ (equation 1.2). If we write equation 3.88 on a microscopic piece of cross-sectional area in a pipe flow with varying $\underline{v}$ and integrate over the pipe cross section (recall equation 1.158) we obtain equation 1.2; this calculation is shown in Chapter 6. In the example below, we practice a bit with the relations we have just developed.

EXAMPLE 3.8 *Consider a control volume in the shape of the square pyramid as shown in Figure 3.28. The square pyramid is a pentahedron with a square for a base and four triangles for sides; the one in Figure 3.28 has four equilateral triangles for sides (a Johnson solid). The pyramid is a control volume placed in a uniform flow (velocity $\underline{v}$ in the flow is constant at every position in space). The flow direction is parallel at all points to a vector in the plane of the pyramid's base that bisects two opposite sides of the base. Calculate the mass flow rate of fluid of density ρ through each of the five sides of the pentahedron. Write your answer in terms of the speed of the fluid v and the pyramid edge-length α.*

SOLUTION The use of a pentahedron as a control volume is unusual, but the calculations involved in solving this problem are not unusual at all when making calculations of the convective contribution to the momentum balance. This problem provides us with an opportunity to practice with angles, geometry, the dot product, and the relations in this section.

The mass flow through a surface is given by equation 3.91.

$$\begin{array}{c}\text{Mass flow of liquid}\\ \text{through surface } A\end{array} \qquad m = \rho\ (\hat{n}\cdot\underline{v})\ A \tag{3.97}$$

For each of the five surfaces of the control volume we need the unit normal $\hat{n}$ and the area A. The density ρ is constant, and the velocity vector $\underline{v}$ is the same at all locations for uniform flow.

We choose as our coordinate system a Cartesian coordinate system with the flow direction as the z-direction.

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} = v\hat{e}_z \tag{3.98}$$

The outwardly pointing unit normal vectors for each surface of the control volume are shown in Figure 3.28. For the bottom of the pyramid, the outwardly pointing unit vector $\underline{a}$ points downward, $\underline{a} = -\hat{e}_x$. The dot product of $\underline{a}$ and $\underline{v} = v\hat{e}_z$ is therefore zero, and the mass flow rate through the bottom is zero:

$$\begin{aligned}
m &= \rho\ (\hat{n}\cdot\underline{v})\ A && (3.99)\\
m|_a &= \rho(\underline{a}\cdot\underline{v})\alpha^2 && (3.100)\\
&= \rho\alpha^2 \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}_{xyz} \cdot \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} && (3.101)\\
&= 0 && (3.102)
\end{aligned}$$

For surface b, the geometry in the inset of Figure 3.28 shows us that the outwardly

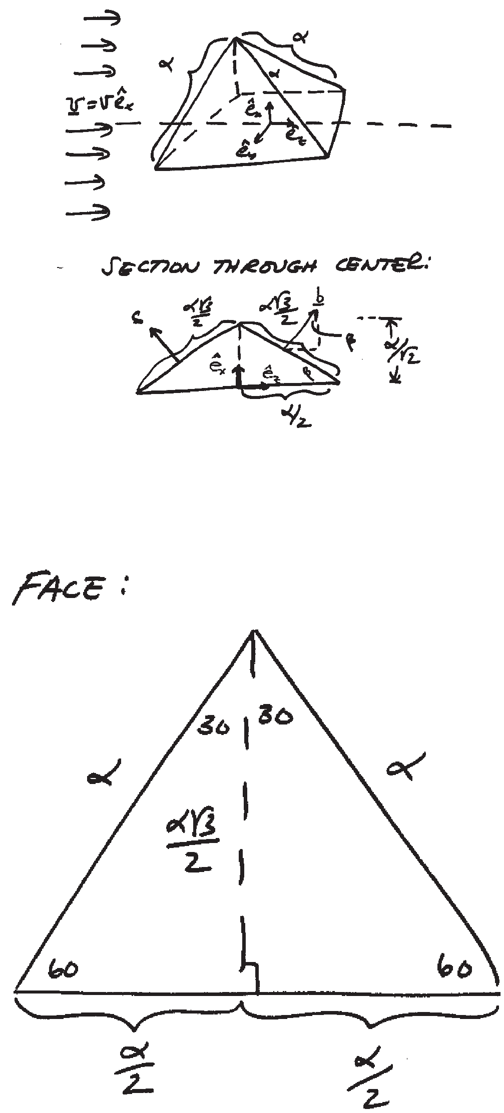

Figure 3.28: The control volume is a square pyramid, which has five sides, four of which are equilateral triangles.

pointing unit normal vector $\underline{b}$ is

$$\hat{n}\big|_b \equiv \underline{b} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{pmatrix}_{xyz} \tag{3.103}$$

and the area of the equilateral triangle that makes up the face is $A = (1/2)(\alpha)(\alpha\sqrt{3}/2)$. The mass flow rate through surface b is therefore

$$\begin{aligned} m &= \rho\ (\hat{n}\cdot\underline{v})\ A & (3.104)\\ m\big|_b &= \rho(\underline{b}\cdot\underline{v})\frac{\alpha^2\sqrt{3}}{4} & (3.105)\\ &= \frac{\rho\alpha^2\sqrt{3}}{4}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}_{xyz}\cdot\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} & (3.106)\\ &= \frac{\rho v\alpha^2}{2\sqrt{2}} & (3.107) \end{aligned}$$

For surface c, the outwardly pointing unit normal vector $\underline{c}$ is similar to $\underline{b}$, but the z-component points in the opposite direction.

$$\hat{n}\big|_c \equiv \underline{c} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\sqrt{\frac{2}{3}} \end{pmatrix}_{xyz} \tag{3.108}$$

The mass flow rate through surface c is therefore

$$\begin{aligned} m\big|_c &= \rho(\underline{c}\cdot\underline{v})\frac{\alpha^2\sqrt{3}}{4} & (3.109)\\ &= \frac{\rho\alpha^2\sqrt{3}}{4}\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}_{xyz}\cdot\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}_{xyz} & (3.110)\\ &= -\frac{\rho v\alpha^2}{2\sqrt{2}} & (3.111) \end{aligned}$$

The mass flow rates out through surfaces b and c are the same but one is positive, indicating that the flow is outwards (surface $\underline{b}$) and one is negative, indicating that the flow is inwards (surface c).

For surfaces d and h, the unit normal vectors are in the xy-plane, and thus when the outwardly pointed unit normal $\hat{n}$ is dotted with $\underline{v} = v\hat{e}_z$ in each case, we get zero; there is no mass flow out of the control volume through either of these surfaces.

$$\hat{d} = \begin{pmatrix} d_x \\ d_y \\ 0 \end{pmatrix}_{xyz} \tag{3.112}$$

$$\hat{d} \cdot \underline{v} = (d_x \hat{e}_x + d_y \hat{e}_y) \cdot v\hat{e}_z = 0 \tag{3.113}$$

$$\hat{h} = \begin{pmatrix} h_x \\ h_y \\ 0 \end{pmatrix}_{xyz} \tag{3.114}$$

$$\hat{h} \cdot \underline{v} = (h_x \hat{e}_x + h_y \hat{e}_y) \cdot \underline{v}\hat{e}_z = 0 \tag{3.115}$$

Finally, notice that the sum of all the mass flow rates is zero; this is in accord with the mass balance that at steady state the net outflow of mass from the control volume is zero.

$$\begin{pmatrix} \text{net outflow} \\ \text{of mass from} \\ \text{control volume (CV)} \end{pmatrix} = m|_a + m|_b + m|_c + m|_d + m|_h \tag{3.116}$$

$$= 0 + \frac{\rho v \alpha^2}{2\sqrt{2}} - \frac{\rho v \alpha^2}{2\sqrt{2}} + 0 + 0 \tag{3.117}$$

$$= 0 \tag{3.118}$$

We return now to our work with equation 3.67. We seek to covert the two word-expressions in that equation to mathematical terms. Both of the word-expressions under consideration account for momentum flows of fluid through the surfaces that bound the control volume. In the previous example we practiced writing momentum flows through a surface (equation 3.96), and we now turn to applying this technique to the control volume.

Beginning with the blue fluid that enters the control volume, consider the surface area S_{in} through which blue fluid enters (Figure 3.29). We have chosen a surface with an arbitrary shape and orientation for this derivation. In a general flow, fluid velocity varies with position, and therefore some care must be taken when calculating the momentum entering the control volume through S_{in}. We must divide up the surface S_{in} in some way and sum the contributions from various regions. In addition, the surface S_{in} is not generally flat, and therefore the task of dividing S_{in} is itself a challenge. This very problem has been addressed in the development of integral calculus (appendix C.2.1), and we can directly apply these methods to the calculation of the flow of momentum through S_{in}.

Our approach is to project S_{in} onto a plane we arbitrarily call the xy-plane (Figure 3.30). The area of the projection is $\mathcal{R}$. Since $\mathcal{R}$ is in the xy-plane, the unit normal to $\mathcal{R}$ is $\hat{e}_z$. We divide the projection $\mathcal{R}$ into areas $\Delta A = \Delta x \Delta y$ and seek to write the momentum