Final Exam Formulas

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Rate of deformation tensor: $\dot{\underline{\underline{\boldsymbol{\gamma}}}} = \nabla \underline{\boldsymbol{v}} + (\nabla \underline{\boldsymbol{v}})^T$

Rate of deformation: $\dot{\boldsymbol{\gamma}} = \left| \dot{\underline{\underline{\boldsymbol{\gamma}}}} \right|$

Tensor magnitude: $\boldsymbol{A} = \left| \underline{\underline{\boldsymbol{A}}} \right| = +\sqrt{\frac{\underline{\underline{\boldsymbol{A}}}:\underline{\underline{\boldsymbol{A}}}}{2}}$

Shear strain: $\boldsymbol{\gamma_{21}(t_a, t_b)} = \int_{t_a}^{t_b} \dot{\boldsymbol{\gamma}}_{21}(\boldsymbol{t''}) \boldsymbol{dt''}$

Navier-Stokes Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$

Cauchy Momentum Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$

Continuity Equation: $\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \underline{v}\right)$

Fluid force $\underline{\boldsymbol{F}}$ on a surface S:

$$\underline{\boldsymbol{F}} = \iint_S \left[\hat{\boldsymbol{n}} \cdot -\underline{\underline{\boldsymbol{\Pi}}}\right]\Big|_{surface} \boldsymbol{dA}$$

Flow rate $\boldsymbol{Q}$ through a surface S:

$$\boldsymbol{Q} = \iint_S \left[\hat{\boldsymbol{n}} \cdot \underline{\boldsymbol{v}}\right]_{surface} \boldsymbol{dA}$$

Fluid torque $\underline{\boldsymbol{T}}$ on a surface S: ($\underline{\boldsymbol{R}}$ is the vector from the axis of rotation to the point of application of the force)

$$\underline{\boldsymbol{T}} = \iint_S \left[\underline{\boldsymbol{R}} \times \left(\hat{\boldsymbol{n}} \cdot -\underline{\underline{\boldsymbol{\Pi}}}\right)\right]_{surface} \boldsymbol{dA}$$

Newtonian, incompressible fluid: $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{\mu}(\nabla \underline{\boldsymbol{v}} + (\nabla \underline{\boldsymbol{v}})^{\boldsymbol{T}})$

Hookean solid (small strain): $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{G}\underline{\underline{\boldsymbol{\gamma}}}(\boldsymbol{t}, \boldsymbol{t}')$

Generalized Newtonian fluid (GNF): $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}})\underline{\underline{\dot{\boldsymbol{\gamma}}}}$

Power-law GNF model: $\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}}) = \boldsymbol{m}\dot{\boldsymbol{\gamma}}^{\boldsymbol{n}-\mathbf{1}}$

(Note that m and n are parameters of the model and are constants)

Carreau-Yasuda GNF model: $\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}}) = \boldsymbol{\eta}_{\infty} + (\boldsymbol{\eta}_{\mathbf{0}} - \boldsymbol{\eta}_{\infty})[\mathbf{1} + (\dot{\boldsymbol{\gamma}}\boldsymbol{\lambda})^{\boldsymbol{a}}]^{\frac{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{a}}}$

(Note that a, λ and n, η_0, and η_∞ are parameters of the model and are constants)

Generalized Linear Viscoelastic Model (GLVE) (rate version): $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = -\int_{-\infty}^{\boldsymbol{t}} \boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}')\underline{\underline{\dot{\boldsymbol{\gamma}}}}(\boldsymbol{t}')\boldsymbol{d}\boldsymbol{t}'$

Generalized Linear Viscoelastic Model (GLVE) (strain version) : $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = +\int_{-\infty}^{\boldsymbol{t}} \frac{\partial \boldsymbol{G}(\boldsymbol{t}-\boldsymbol{t}')}{\partial \boldsymbol{t}'}\underline{\underline{\boldsymbol{\gamma}}}(\boldsymbol{t}, \boldsymbol{t}')\boldsymbol{d}\boldsymbol{t}'$

Maxwell GLVE model: $\boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}') = \frac{\boldsymbol{\eta}_{\mathbf{0}}}{\boldsymbol{\lambda}}\boldsymbol{e}^{-(\boldsymbol{t}-\boldsymbol{t}')/\boldsymbol{\lambda}}$

Generalized Maxwell GLVE model: $\boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}') = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{N}} \frac{\boldsymbol{\eta}_{\boldsymbol{k}}}{\boldsymbol{\lambda}_{\boldsymbol{k}}}\boldsymbol{e}^{-(\boldsymbol{t}-\boldsymbol{t}')/\boldsymbol{\lambda}_{\boldsymbol{k}}}$

Elongational flow (uniaxial, biaxial): $\underline{\boldsymbol{v}} = \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123}$

Shear flow: $\underline{\boldsymbol{v}} = \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$

Steady shearing kinematics: $\dot{\varsigma}(t) = \dot{\gamma}_0$ for all values of time t

Start-up of steady shearing kinematics: $\dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$

Cessation of steady shearing kinematics: $\dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$

Step shear strain kinematics: $\dot{\varsigma}(t) = lim_{\epsilon \to 0} \begin{cases} 0 & t < 0 \\ \gamma_0/\epsilon & 0 \leq t < \epsilon \\ 0 & 0 \leq t \end{cases}$

Steady elongational kinematics: $\dot{\varepsilon}(t) = \dot{\varepsilon}_0$

Start-up of steady elongation kinematics: $\dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases}$

Cessation of steady elongation kinematics: $\dot{\varepsilon}(t) = \begin{cases} \dot{\varepsilon}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$

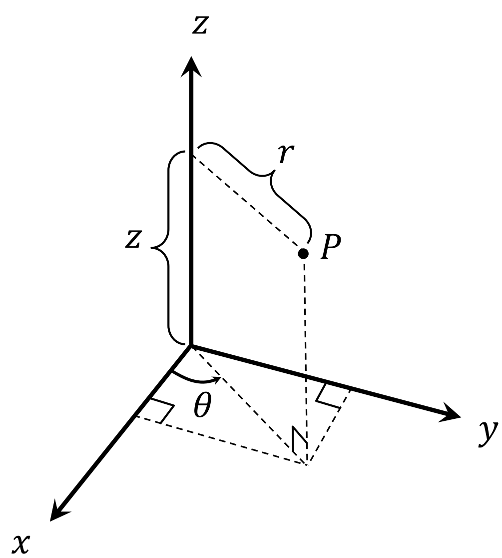
Shear viscosity: $\eta = \frac{-(\tau_{21})}{\dot{\gamma}_0}$

Shear normal stress coefficients: $\Psi_1 = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}, \Psi_2 = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0}$

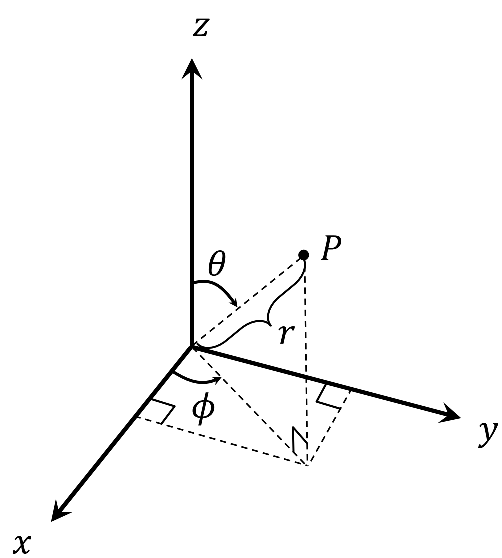
Shear relaxation modulus (step shear strain): $G(t, \gamma_0) = \frac{-\tau_{21}(t)}{\gamma_0}$

Elongational viscosity: $\bar{\eta} = \eta_e = \frac{-(\tau_{33} - \tau_{11})}{\dot{\varepsilon}_0}$

Cylindrical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings around the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate is perpendicular to the z-axis.



Spherical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings down from the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate emits radially from the origin; these are different from their definitions in the cylindrical system above.



System	**Coordinates**	**Basis vectors**
Spherical	$\boldsymbol{x = r\sin\theta\cos\phi}$	$\hat{\boldsymbol{e}}_r = (\sin\theta\cos\phi)\hat{\boldsymbol{e}}_x + (\sin\theta\sin\phi)\hat{\boldsymbol{e}}_y + \cos\theta\hat{\boldsymbol{e}}_z$
Spherical	$\boldsymbol{y = r\sin\theta\sin\phi}$	$\hat{\boldsymbol{e}}_\theta = (\cos\theta\cos\phi)\hat{\boldsymbol{e}}_x + (\cos\theta\sin\phi)\hat{\boldsymbol{e}}_y + (-\sin\theta)\hat{\boldsymbol{e}}_z$
Spherical	$\boldsymbol{z = r\cos\theta}$	$\hat{\boldsymbol{e}}_\phi = (-\sin\phi)\hat{\boldsymbol{e}}_x + \cos\phi\,\hat{\boldsymbol{e}}_y$
Cylindrical	$\boldsymbol{x = r\cos\theta}$	$\hat{\boldsymbol{e}}_r = \cos\theta\,\hat{\boldsymbol{e}}_x + \sin\theta\,\hat{\boldsymbol{e}}_y$
Cylindrical	$\boldsymbol{y = r\sin\theta}$	$\hat{\boldsymbol{e}}_\theta = (-\sin\theta)\hat{\boldsymbol{e}}_x + \cos\theta\,\hat{\boldsymbol{e}}_y$
Cylindrical	$\boldsymbol{z = z}$	$\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_z$

Differential Areas and Volumes

Coordinate system	***Surface Differential dS***
Cartesian (top, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_z$)	$\boldsymbol{dS = dxdy}$
Cartesian (top, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_y$)	$\boldsymbol{dS = dxdz}$
Cartesian (top, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_x$)	$\boldsymbol{dS = dydz}$
Cylindrical (top, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_z$)	$\boldsymbol{dS = rdrd\theta}$
Cylindrical (side, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_r$)	$\boldsymbol{dS = Rd\theta dz}$
Spherical (at $r = R, \hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_r$)	$\boldsymbol{dS = R^2 \sin\theta d\theta d\phi}$

Coordinate system	***Volume Differential dV***
Cartesian	$\boldsymbol{dV = dxdydz}$
Cylindrical	$\boldsymbol{dV = rdrd\theta dz}$
Spherical	$\boldsymbol{dV = r^2 \sin\theta drd\theta d\phi}$

Tensor Invariants (p40)

$$\mathbf{I}_{\underline{\underline{\mathbf{B}}}} \equiv \sum_{i=1}^{3} B_{ii} = trace(\underline{\underline{B}})$$

$$\mathbf{II}_{\underline{\underline{\mathbf{B}}}} \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij} B_{ji} = trace(\underline{\underline{B}} \cdot \underline{\underline{B}})$$

$$\mathbf{III}_{\underline{\underline{\mathbf{B}}}} \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} B_{ij} B_{jk} B_{ki} = trace(\underline{\underline{B}} \cdot \underline{\underline{B}} \cdot \underline{\underline{B}})$$

Note that $\left|\underline{\underline{B}}\right| = +\sqrt{\frac{II_{\underline{\underline{B}}}}{2}}$

Table of Integrals

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad \boldsymbol{n} \text{ is a constant}$$

$$\int e^u du = e^u + C$$

$$\int ue^u du = e^u(u-1) + C$$

$$\int u^2 e^u du = e^u(u^2 - 2u + 2) + C$$

$$\int \cos(u)\, du = \sin(u) + C$$

$$\int \sin(u)\, du = -\cos(u) + C$$

$$\int u\cos(u) du = u\sin(u) + \cos(u) + C$$

$$\int u\sin(u)\, du = \sin(u) - u\cos(u) + C$$