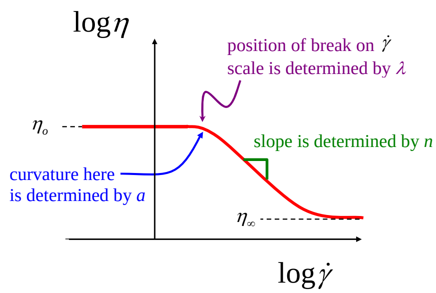
Chapter 7: Generalized Newtonian fluids

CM4650
Polymer Rheology
Michigan Tech

Carreau-Yassuda GNF



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Back to our main goal:
Constitutive Equation – an accounting for <u>all</u> stresses, all flows

Newtonian fluids:
(all flows)

stress tensor

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$$

Rate-of-deformation <u>tensor</u>

In general:

$$\underline{\underline{\tau}} = -f\left(\underline{\underline{\dot{\gamma}}}\right)$$

In the general case, f needs to be a non-linear function (in time and position)

What should we choose for the function *f*?

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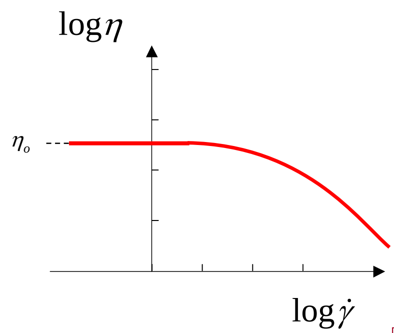
Non-Newtonian, Inelastic Fluids

First, we concentrate on the observation that **shear viscosity depends on shear rate**.



$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

Non-Newtonian viscosity, η

$$\dot{\gamma} \equiv \left| \frac{\partial v_1}{\partial x_2} \right|$$

shear rate

We will design a constitutive equation that predicts this behavior in shear flow

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Newtonian Constitutive Equation

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$$

For Newton's experiment (shear flow): $\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = -\mu \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

We could make this equation give the right answer (shear thinning) in steady shear flow if we substituted ***a function of shear rate*** for the constant viscosity.

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Generalized Newtonian Fluid (GNF)
constitutive equation

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = -\eta(\dot{\gamma}) \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

SHEAR FLOW

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\gamma} \equiv \left| \frac{\partial v_1}{\partial x_2} \right|$$

GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

ALL FLOWS

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \qquad \dot{\gamma} \equiv \left| \dot{\underline{\underline{\gamma}}} \right|$$

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Constitutive Equation – **an accounting for all stresses, all flows**

$$\underline{\underline{\tau}} = -f\left(\dot{\underline{\underline{\gamma}}}\right)$$

A simple choice for *f*:

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\dot{\underline{\underline{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

$$\dot{\gamma} = \left|\dot{\underline{\underline{\gamma}}}\right| \equiv \left| \sqrt{\frac{1}{2} \dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}} \right|$$

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$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

What do we pick for $\eta(\dot{\gamma})$?

- Something that matches the data;
- Something simple, so that the calculations are easy

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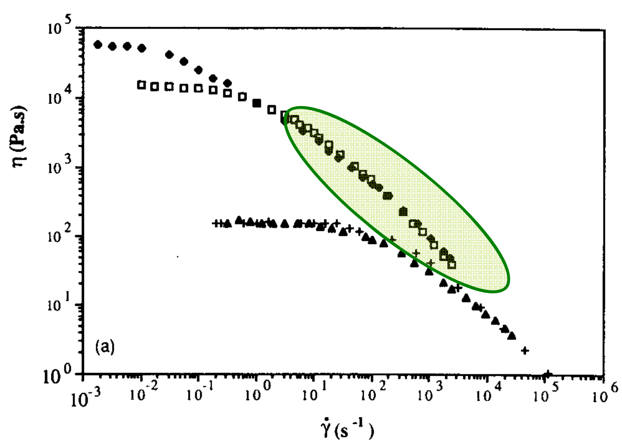
In processing, the high-shear-rate behavior is the most important.



+ linear 131 kg/mole
▲ branched 156 kg/mole
□ linear 418 kg/mol
◆ branched 428 kg/mol

Figure 6.3, p. 172 Piau et al., linear and branched PDMS

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Power-law model for viscosity

$$\eta = m\dot{\gamma}^{n-1}$$

in shear flow $\dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right|$

$$\eta = m\left|\frac{dv_1}{dx_2}\right|^{n-1}$$

(in shear flow)

On a log-log plot, this would give a straight line:

$$\underbrace{\log\eta}_{Y} = \underbrace{\log m}_{B} + \underbrace{(n-1)}_{M}\underbrace{\log\left|\frac{dv_1}{dx_2}\right|}_{X}$$

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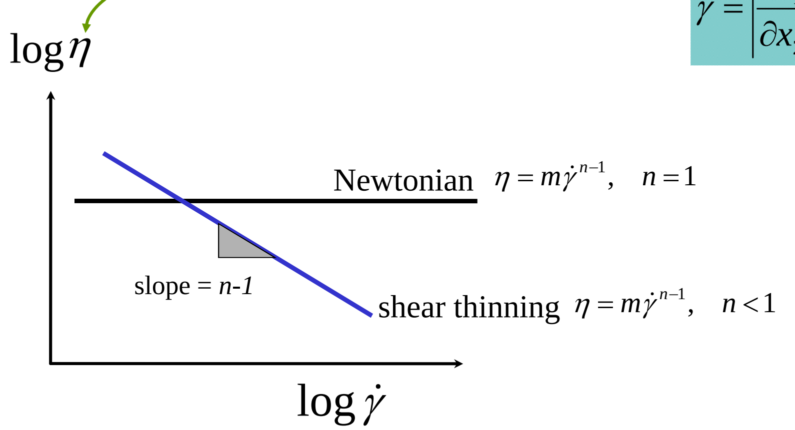
Power-law model for viscosity

steady shear flow

Non-Newtonian shear viscosity

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

$$\dot{\gamma} = \left|\frac{\partial v_1}{\partial x_2}\right|$$



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Power-Law Generalized Newtonian Fluid

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta = -m\dot{\gamma}^{n-1}$$

m or *K* = consistency index (*m* = μ for Newtonian)
n = power-law index (*n* = 1 for Newtonian)

$$\dot{\gamma} \equiv \left|\underline{\underline{\dot{\gamma}}}\right|$$

(Usually 0.5≤ *n* ≤1)

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Carreau-Yassuda GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

A model with 5 parameters

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + (\dot{\gamma}\lambda)^a\right]^{\frac{n-1}{a}}$$

- The viscosity function approaches the constant value of η_∞ as deformation rate get large
- The viscosity function approaches the constant value η_0 as deformation rate gets small
- λ is the time constant for the fluid
- *n* determines the slope of the power-law region

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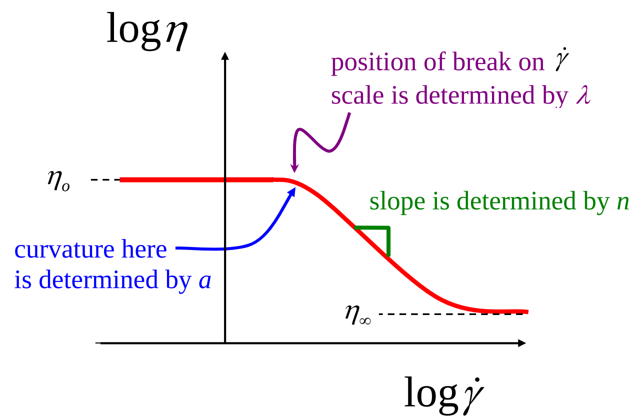
Carreau-Yassuda GNF



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What about shear thickening?

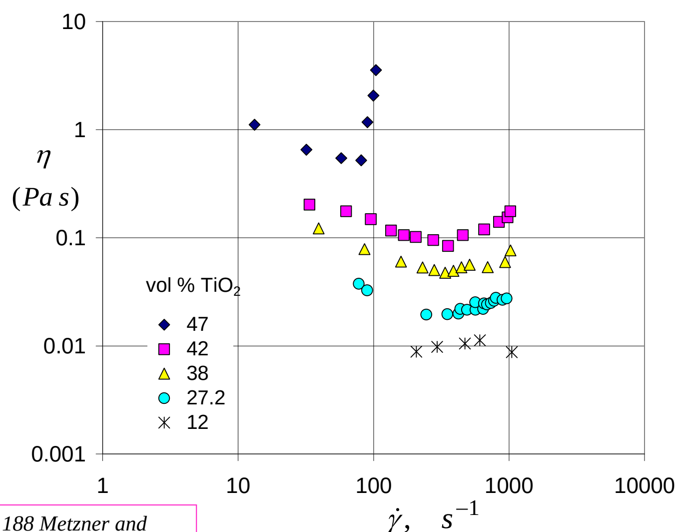

Figure 6.27, p. 188 Metzner and Whitlock; TiO_2/water suspensions

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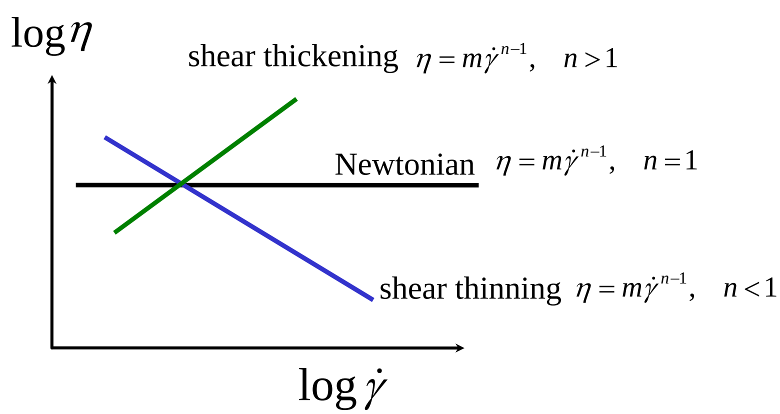
Power-Law GNF

steady shear flow



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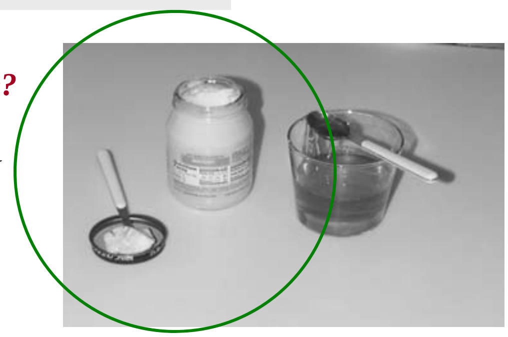
Other Inelastic Fluids

What about mayonnaise?



Mayonnaise and many other like fluids (paint, ketchup, most suspensions, asphalt) is able to sustain a **yield stress.**

Once the fluid begins to deform under an imposed stress, the viscosity may either be constant or may shear-thin. This type of steady shear viscosity behavior can be modeled with a GNF.

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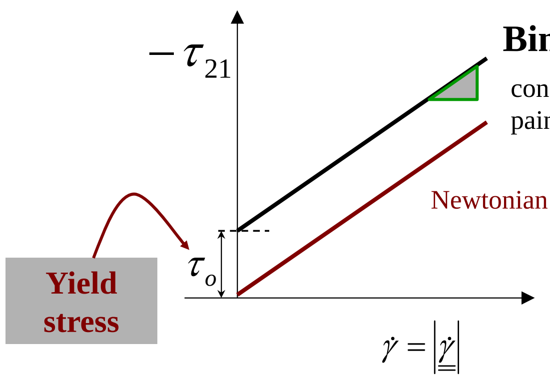
Non-Newtonian Fluids

For some fluids, no flow occurs when moderate stresses are applied.

Bingham plastic

constant slope = μ_o (mayo, paints, suspensions)



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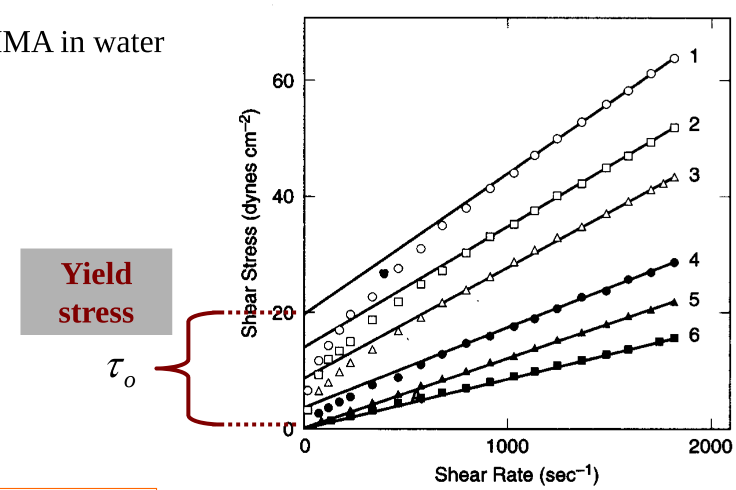
Non-Newtonian Fluids

Bingham plastic

PMMA in water



Friend and Hunter, 1971; dispersions of PMMA in water at various ζ-potentials; From Larson, p353.

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Bingham Plastic
steady shear flow

Non-Newtonian viscosity, η

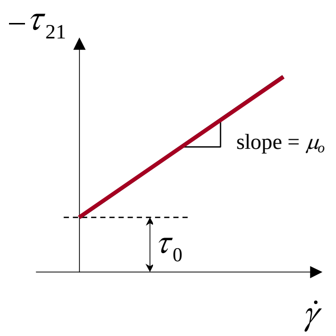
$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

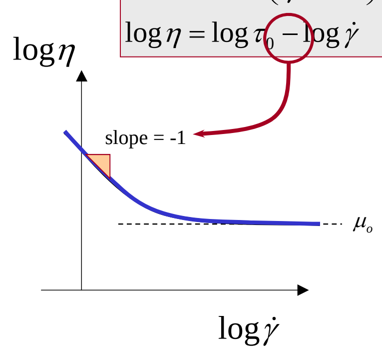
$$\eta = \frac{\tau_0}{\dot{\gamma}} + \mu_0$$

$$\lim_{\dot{\gamma}\to 0}(\eta) = \lim_{\dot{\gamma}\to 0}\left(\frac{\tau_0}{\dot{\gamma}} + \mu_0\right) = \frac{\tau_0}{\dot{\gamma}}$$

$$\log\eta = \log\tau_0 - \log\dot{\gamma}$$

$$-\tau_{21} \equiv \tau_0 + \mu_0\dot{\gamma}$$





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Bingham GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

A model with 2 parameters

$$\eta(\dot{\gamma}) = \begin{cases} \infty & \left|\underline{\underline{\tau}}\right| \le \tau_0 \\ \mu_o + \dfrac{\tau_0}{\dot{\gamma}} & \left|\underline{\underline{\tau}}\right| > \tau_0 \end{cases}$$

μ_o = viscosity parameter
τ_y = yield stress

There is no flow until the shear stress exceeds a critical value τ_0 called the yield stress.

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Other GNF viscosity models

See Carreau, DeKee, and Chhabra for complete discussion (*Rheology of Polymeric Systems*, Hanser, 1997)

Ellis Model $\eta = \dfrac{\eta_0}{1 + \left|\dfrac{\tau}{\tau_0}\right|^{\alpha-1}}$ $\tau = |\underline{\underline{\tau}}|$

4-Parameter Carreau Model (same as CY with a=2)

Cross-Williamson Model (same as CY with a=1, $\eta_\infty = 0$)

DeKee Model $\eta = \eta_1 e^{-\lambda\dot{\gamma}} + \eta_2 e^{-0.1\lambda\dot{\gamma}} + \eta_\infty$

Casson Model $\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{\eta_0 \gamma}$ $\tau = |\underline{\underline{\tau}}|$

Herschel-Bulkley Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + m\dot{\gamma}^{n-1}$

DeKee-Turcotte Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + \eta_1 e^{-\lambda\dot{\gamma}}$

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

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Yield stress plus power-law viscosity behavior

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What now?

•Predict material functions with the Generalized Newtonian Constitutive Equation.

Example: Elongational viscosity, etc.

•Calculate velocity and stress fields predicted by Generalized Newtonian Constitutive Equations

Example: Poiseuille flow, drag flow, etc.

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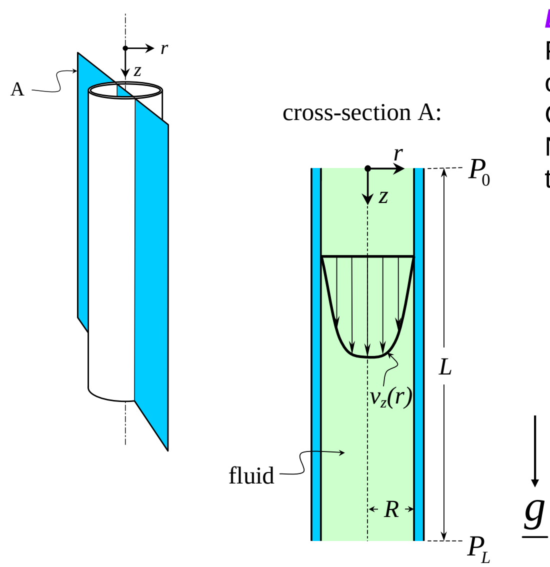

EXAMPLE:
Pressure-driven flow of a Power-Law Generalized Newtonian fluid in a tube

- •steady state
- •well developed
- •long tube

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Velocity field
Poiseuille flow of a power-law fluid:

$$v_z(r) = \left(\frac{R\left(L\rho g + P_o - P_L\right)}{2Lm} \right)^{\frac{1}{n}} \left(\frac{R}{\frac{1}{n}+1} \right) \left(1 - \left(\frac{r}{R} \right)^{\frac{1}{n}+1} \right)$$

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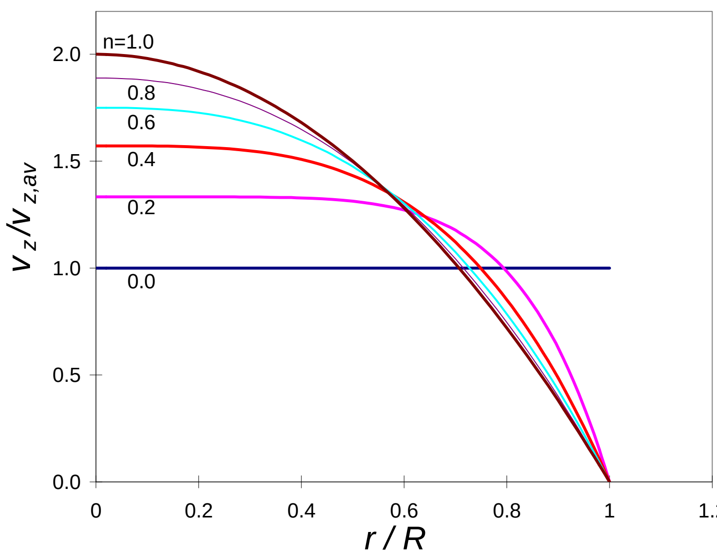
Solution to Poiseuille flow in a tube
incompressible, power-law fluid



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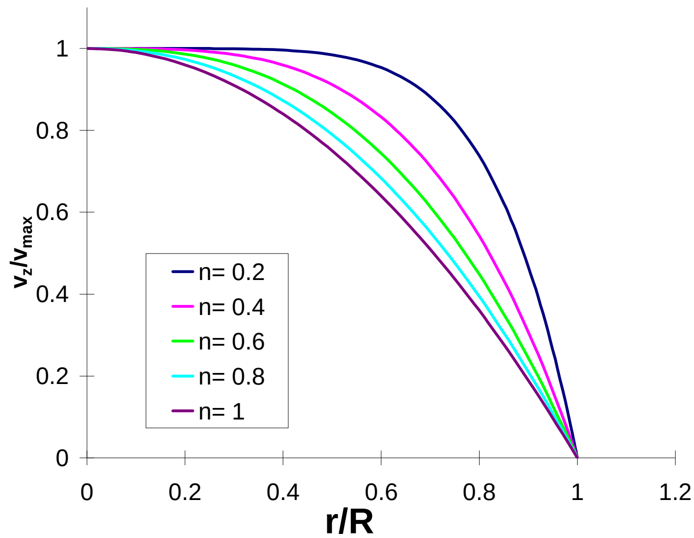
Solution to Poiseuille flow in a tube

incompressible, power-law fluid



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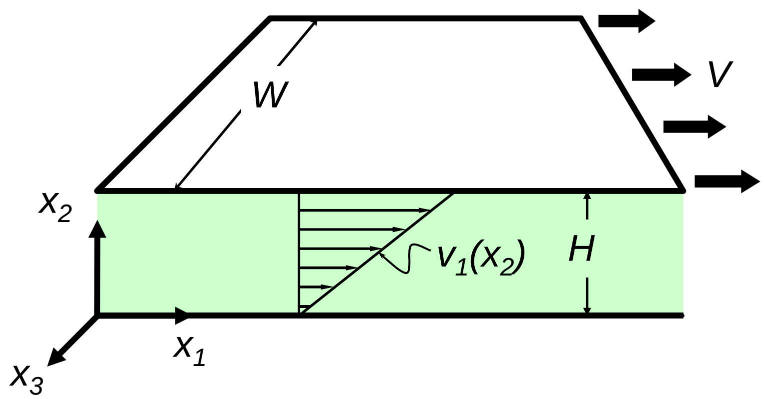
EXAMPLE: Drag flow of a Power-Law GNF between infinite parallel plates

- steady state
- incompressible fluid
- infinitely wide, long



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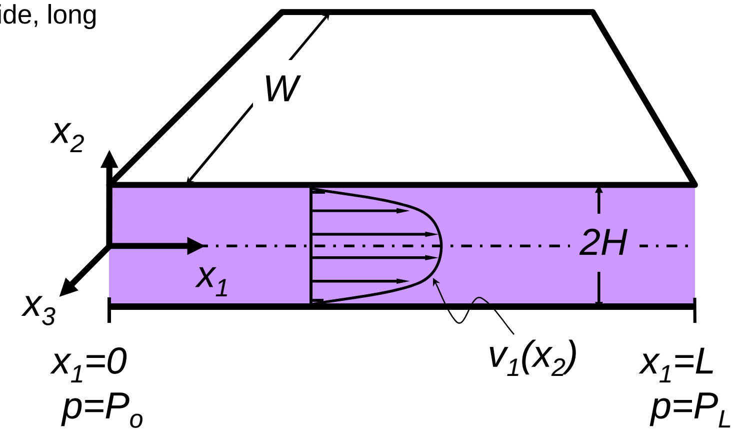
EXAMPLE: **Pressure-driven flow of a Power-Law GNF between infinite parallel plates**

- steady state
- incompressible fluid
- infinitely wide, long



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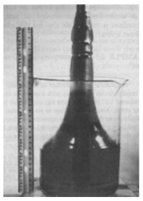
The steady shear viscosity function η can be fit to experimental data to an arbitrarily high precision.

Does this mean that *Generalized Newtonian Fluid* models are okay to use in all situations?

Not necessarily. A constitutive model needs to be able to predict *all stresses* in *all flows*, not just shear stresses in steady shearing. We need to check predictions.

For example, does the GNF predict the shear normal stresses?

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$



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Generalized Newtonian Fluid (GNF) constitutive equation

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2}+\frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

In Shear Flow:

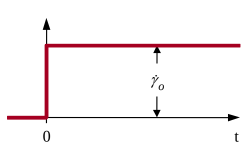
$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right| \qquad \underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = \begin{pmatrix} 0 & -\eta(\dot{\gamma})\frac{\partial v_1}{\partial x_2} & 0 \\ -\eta(\dot{\gamma})\frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

No matter what we pick for the function $\eta(\dot{\gamma})$, **we cannot predict** shear normal stresses with a Generalized Newtonian Fluid.

What does the GNF predict for start-up shear stresses?

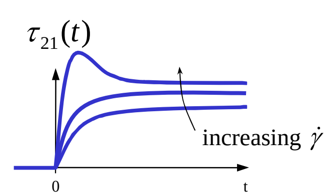
imposed shear rate

$$\dot{\gamma}_{21} = v_1(t)/H$$

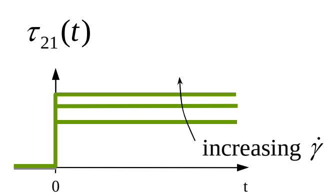


shear stress response

What the data show:

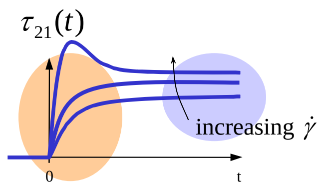


What the GNF models predict:



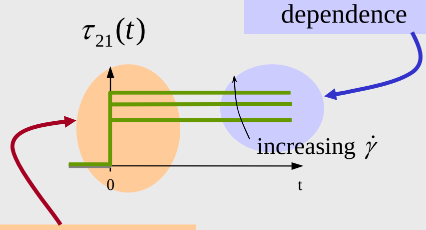
Start-up shear stresses

What the data show:



What the **GNF** models predict:

Correctly captures rate dependence



misses start-up effects

No matter what we pick for the function $\eta(\dot{\gamma})$, **we cannot predict** the time-dependence of shear start-up correctly with a GNF.

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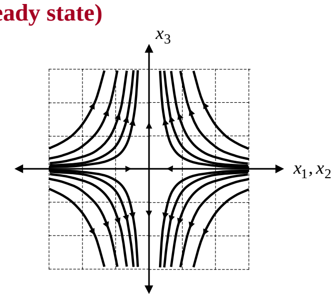
What does the GNF predict in steady elongational flow?

imposed deformation
(steady state)



elongational stress response

What the data show:

$$\lim_{\dot{\varepsilon}\to 0} \bar{\eta} = 3\eta_0$$

Trouton's Rule

(there is limited elongational viscosity data available)

What the GNF models predict:

$$\bar{\eta} = 2\eta$$

For all deformation rates

If a material shear-thins, GNF predicts it will tension-thin.

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Summary: *Generalized Newtonian Fluid Constitutive Equations*

PRO:

- A first constitutive equation
- Can match steady shearing data very well
- Simple to calculate with
- Found to predict pressure-drop/flow rate relationships well

CON:

- Fails to predict shear normal stresses
- Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
- Derived ad hoc from shear observations; unclear of validity in non-shear flows

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Summary: *Generalized Newtonian Fluid Constitutive Equations*

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We now look to address this failing of GNF models by seeking to incorporate *memory*.

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Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info})$$

The stress expression:

- ***Must be of tensor order***
- ***Must be a tensor (independent of coordinate system)***
- ***Must be a symmetric tensor***
- ***Must make predictions that are independent of the observer***
- ***Should correctly predict observed flow/deformation behavior***

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Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{m}$$

Tensor invariants – scalars associated with a tensor that do not depend on coordinate system

The stress expression:

- ***Must be of tensor order***
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Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace \underline{\underline{A}} = tr \underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace \underline{\underline{A}} = \sum_{p=1}^{3} A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = \sum_{p=1}^{3} \sum_{k=1}^{3} A_{pk} A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \sum_{p=1}^{3} \sum_{j=1}^{3} \sum_{h=1}^{3} A_{pj} A_{jh} A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

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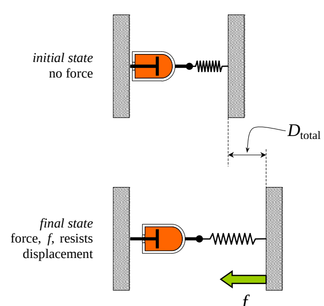
Chapter 8: Memory Effects: GLVE

CM4650
Polymer Rheology
Michigan Tech

Maxwell's model combines viscous and elastic responses in series

Spring (elastic) and dashpot (viscous) in series:



Displacements are additive:

$$D_{total} = D_{spring} + D_{dashpot}$$

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