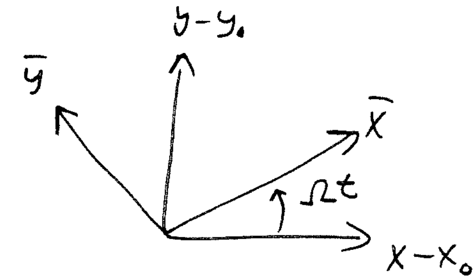
9.48 Show that the Lodge eqn is frame invariant using the turntable example.

SOLN

For the example at the end of Chapter 7 we considered a steady shear flow with respect to two different coordinate systems, one stationary (x, y, z) and one rotating $(\bar{x}, \bar{y}, \bar{z})$.



$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 \bar{y} \\ 0 \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}}$$

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 \cos\Omega t \left[(y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t \right] - \Omega (y-y_0) \\ \dot{\gamma}_0 \sin\Omega t \left[(y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t \right] + \Omega (x-x_0) \\ 0 \end{pmatrix}_{xyz}$$

Lodge eqn:

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} \underline{\underline{C}}^{-1}(t',t)\, dt'$$

We need to calculate the Finger tensor in the rotating frame.

$$\underline{\underline{C}}^{-1}(t',t) = (\underline{\underline{F}}^{-1})^T \cdot \underline{\underline{F}}^{-1}$$

$$\underline{\underline{F}}^{-1} = \frac{\partial \underline{r}}{\partial \underline{r}'} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz}$$

We have from the practice plm

(736)
$$\bar{x} = (y-y_0)\sin\Omega t + (x-x_0)\cos\Omega t$$
$$\bar{y} = (y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t$$
$$\bar{z} = z$$

We know that in the rotating frame, the flow is steady shear:

(1) $\bar{x} = \bar{x}' + \dot{\gamma}_o \bar{y}(t-t')$

(2) $\bar{y} = \bar{y}'$

(3) $\bar{z} = \bar{z}'$

where $\bar{x}' = \bar{x}(t')$
$\bar{y}' = \bar{y}(t')$
$\bar{z}' = \bar{z}(t')$

Substituting $\bar{x}(x,y,z)$ and $\bar{y}(x,y,z)$:

(1)' $(y-y_o)\sin\Omega t + (x-x_o)\cos\Omega t$

$$= (y'-y_o)\sin\Omega t' + (x'-x_o)\cos\Omega t' + \dot{\gamma}_o\left[(y-y_o)\cos\Omega t - (x-x_o)\sin\Omega t\right](t-t')$$

(2)' $(y-y_o)\cos\Omega t - (x-x_o)\sin\Omega t = (y'-y_o)\cos\Omega t' - (x'-x_o)\sin\Omega$

(3)' $z = z'$

We need to combine equations (1)' and (2)' to get $y(x'y'z')$ and $x(x'y'z')$

$y(x'y'z')$:

$$(y-y_0)\sin\Omega t + \frac{\cos\Omega t}{\sin\Omega t}\left\{(y-y_0)\cos\Omega t - (y'-y_0)\cos\Omega t' + (x'-x_0)\sin\Omega t'\right\}$$

$$= (y'-y_0)\sin\Omega t' + (x'-x_0)\cos\Omega t' + \dot{\gamma}_0(t-t')\left[(y'-y_0)\cos\Omega t' - (x'-x_0)\sin\Omega t'\right]$$

$\frac{\partial}{\partial y'}$ of this eqn: (x', z' constant)

$$\frac{\partial y}{\partial y'}\sin\Omega t + \frac{\cos\Omega t}{\sin\Omega t}\left\{\cos\Omega t\,\frac{\partial y}{\partial y'} - \cos\Omega t'\right\}$$

$$= \sin\Omega t' + \dot{\gamma}_0(t-t')\cos\Omega t'$$

$$\frac{\partial y}{\partial y'}\left(\sin\Omega t + \frac{\cos^2\Omega t}{\sin\Omega t}\right) = \frac{\cos\Omega t\cos\Omega t'}{\sin\Omega t} + \sin\Omega t' + \dot{\gamma}_0(t-t')\cos\Omega t'$$

$$\boxed{\frac{\partial y}{\partial y'} = \cos\Omega t \cos\Omega t' + \sin\Omega t' \sin\Omega t + \dot{\gamma}_0 (t-t') \cos\Omega t' \sin\Omega t}$$

calculating $x(x', y', z')$:

$$\left[(y'-y_0)\cos\Omega t' - (x'-x_0)\sin\Omega t' + (x-x_0)\sin\Omega t\right]\frac{\sin\Omega t}{\cos\Omega t}$$

$$+(x-x_0)\cos\Omega t = (y'-y_0)\sin\Omega t' + (x'-x_0)\cos\Omega t'$$

$$+\dot{\gamma}_0(t-t')\left[(y'-y_0)\cos\Omega t' - (x'-x_0)\sin\Omega t'\right]$$

$\frac{\partial}{\partial x'}$ of this eqn: (y', z' const)

$$\left[-\sin\Omega t' + \sin\Omega t\frac{\partial x}{\partial x'}\right]\frac{\sin\Omega t}{\cos\Omega t} + \cos\Omega t\frac{\partial x}{\partial x'}$$

$$= \cos\Omega t' + \dot{\gamma}_0(t-t')(-1)\sin\Omega t'$$

$$\frac{\partial X}{\partial X'}\left(\frac{\sin^2\Omega t}{\cos\Omega t}+\cos\Omega t\right)=\frac{\sin\Omega t'\sin\Omega t}{\cos\Omega t}+\cos\Omega t'$$

$$-(t-t')\dot{\gamma}_0\sin\Omega t'$$

$$\boxed{\frac{\partial X}{\partial X'}=\sin\Omega t'\sin\Omega t+\cos\Omega t'\cos\Omega t-\dot{\gamma}_0(t-t')\frac{\sin\Omega t'}{\cos\Omega t}}$$

BACK TO $y(x'y'z')$ relation

$\frac{\partial}{\partial X'}$ of that eqn: (y', z' constant)

$$\frac{\partial y}{\partial X'}\sin\Omega t+\frac{\cos\Omega t}{\sin\Omega t}\left[\cos\Omega t\,\frac{\partial y}{\partial X'}+\sin\Omega t'\right]$$

$$=\cos\Omega t'+\dot{\gamma}_0(t-t')(-1)\sin\Omega t'$$

$$\frac{\partial y}{\partial X'}\left(\sin\Omega t+\frac{\cos^2\Omega t}{\sin\Omega t}\right)=-\frac{\sin\Omega t'\cos\Omega t}{\sin\Omega t}+\cos\Omega t'$$

$$-\dot{\gamma}_0(t-t')\sin\Omega t'$$

$$\boxed{\frac{\partial y}{\partial X'}=-\sin\Omega t'\cos\Omega t+\cos\Omega t'\sin\Omega t-\dot{\gamma}_0(t-t')\sin\Omega t'\sin\Omega t}$$

BACK TO $x(x',y',z')$ relation

$\frac{\partial}{\partial y'}$ of that eqn (x', z' constant)

$$\frac{\sin\Omega t}{\cos\Omega t}\left[\cos\Omega t' + \sin\Omega t \frac{\partial x}{\partial y'}\right] + \cos\Omega t \frac{\partial x}{\partial y'}$$

$$= \sin\Omega t' + \dot{\gamma}_0 (t-t')\cos\Omega t' \Big]$$

$$\frac{\partial x}{\partial y'}\left(\frac{\sin^2\Omega t}{\cos\Omega t} + \cos\Omega t\right) = -\frac{\sin\Omega t\cos\Omega t'}{\cos\Omega t} + \sin\Omega t'$$

$$+ \dot{\gamma}_0 (t-t')\cos\Omega t'$$

$$\boxed{\frac{\partial x}{\partial y'} = -\sin\Omega t\cos\Omega t' + \sin\Omega t'\cos\Omega t + \dot{\gamma}_0 (t-t')\cos\Omega t'\cos\Omega t}$$

$$\frac{\partial x}{\partial z'} = \frac{\partial y}{\partial z'} = 0 = \frac{\partial z}{\partial x'} = \frac{\partial z}{\partial y'}$$

$$\frac{\partial z}{\partial z'} = 1$$

(741)

$$\underline{\underline{F}}^{-1} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & 0 \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

$$\underline{\underline{C}}^{-1} = (\underline{\underline{F}}^{-1})^T \cdot \underline{\underline{F}}^{-1}$$

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} & 0 \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & 0 \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial x}{\partial y'}\right)^2 & \frac{\partial x}{\partial x'}\frac{\partial y}{\partial x'} + \frac{\partial x}{\partial y'}\frac{\partial y}{\partial y'} & 0 \\ \frac{\partial y}{\partial x'}\frac{\partial x}{\partial x'} + \frac{\partial y}{\partial y'}\frac{\partial x}{\partial y'} & \left(\frac{\partial y}{\partial x'}\right)^2 + \left(\frac{\partial y}{\partial y'}\right)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

(742)

let $S = \sin \Omega t$ $\quad\quad \gamma = \dot{\gamma}_0 (t - t')$

$S' = \sin \Omega t'$

$C = \cos \Omega t$

$C' = \cos \Omega t'$

$$C_{11}^{-1} = \left(\frac{\partial x}{\partial x'}\right)^2 + \left(\frac{\partial x}{\partial y'}\right)^2$$

$$= (s's + c'c - \gamma s'c)^2 + (-sc' + s'c + \gamma c'c)^2$$

$$\begin{aligned} = & \ s'^2 s^2 + \cancel{c'cs's} - \gamma cs'^2 s \\ & + \cancel{c'cs's} + c'^2 c^2 - \cancel{\gamma c'c^2 s'} \\ & - \gamma s'^2 c s - \cancel{\gamma s' c^2 c'} + \gamma^2 s'^2 c^2 \\ & + s^2 c'^2 - \cancel{sc's'c} - sc'^2 \gamma c \\ & - \cancel{s'csc'} + s'^2 c^2 + \cancel{s'c^2 \gamma c'} \\ & - \gamma c'^2 c s + \cancel{\gamma c' c^2 s'} + \gamma^2 c'^2 c^2 \end{aligned}$$

⋮ ⋮ terms give $\underline{1}$

$$C_{11}^{-1} = 1 - 2\gamma c s'^2 s - 2\gamma c'^2 cs + \gamma^2 c^2$$

$$= 1 - 2\gamma cs (s'^2 + c'^2) + \gamma^2 c^2$$

$$= 1 - 2\gamma cs + \gamma^2 c^2$$

$$C^{-1}_{21} = C^{-1}_{12} = \frac{\partial y}{\partial x'}\frac{\partial x}{\partial x'} + \frac{\partial y}{\partial y'}\frac{\partial x}{\partial y'}$$

$$= (-s'c + c's - \gamma s's)(s's + c'c - \gamma s'c)$$

$$+ (cc' + s's + \gamma c's)(-sc' + s'c + \gamma c'c)$$

$$\begin{aligned}
= \; & -\cancel{s'^2cs} - \cancel{s'c^2c'} + s'^2c^2\gamma \\
& + \cancel{c's^2s'} + \cancel{c'^2sc} - \cancel{c's\gamma s'c} \\
& - \gamma s'^2s^2 - \gamma\cancel{s'sc'c} + \gamma^2 s'^2 sc \\
& - \cancel{c'^2cs} + \cancel{c'c^2s'} + c'^2c^2\gamma \\
& - \cancel{s's^2c'} + \cancel{s'^2sc} + \gamma\cancel{s'sc'c} \\
& - \gamma c'^2s^2 + \cancel{\gamma c's}s'c + \gamma^2c'^2sc
\end{aligned}$$

$$= \gamma\left(s'^2c^2 - s'^2s^2 + c'^2c^2 - c'^2s^2\right)$$

$$+ \gamma^2\left(s'^2sc + c'^2sc\right)$$

$$= \gamma s'^2(c^2 - s^2) + \gamma c'^2(c^2 - s^2) + \gamma^2 sc$$

$$\boxed{C^{-1}_{21} = (c^2 - s^2)\gamma + \gamma^2 sc}$$

(744)

$$C^{-1}_{22} = \left(\frac{\partial y}{\partial x'}\right)^2 + \left(\frac{\partial y}{\partial y'}\right)^2$$

$$= (-s'c + c's - \gamma s's)^2 + (cc' + ss' + \gamma c's)^2$$

$$\begin{aligned}
= {} & (+s'^2c^2) - \cancel{s'cc's} + c\gamma s'^2 s \\
& - \cancel{c'ss'c} + (c'^2s^2) - \cancel{\gamma s'c's^2} \\
& + \gamma s'^2 sc - \cancel{\gamma s's^2c'} + \gamma^2 s'^2 s^2 \\
& (c^2c'^2) + \cancel{ss'cc'} + \gamma s c c'^2 \\
& \cancel{ss'cc'} + (s^2s'^2) + \cancel{\gamma c's^2s'} \\
& \gamma c'^2 sc + \cancel{\gamma c's^2s'} + \gamma^2 c'^2 s^2
\end{aligned}$$

( ) terms give 1

$$C^{-1}_{22} = \gamma\,(2css'^2 + 2csc'^2) + \gamma^2(s'^2s^2 + c'^2s^2)$$

$$\boxed{C^{-1}_{22} = 1 + 2cs\gamma + s^2\gamma^2}$$

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 1 - 2cs\gamma + c^2\gamma^2 & (c^2 - s^2)\gamma + sc\gamma^2 & 0 \\ (c^2 - s^2)\gamma + \gamma^2 sc & 1 + 2cs\gamma + s^2\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\underbrace{xyz}_{\text{stationary frame}}}$$

$$c = \cos\Omega t$$
$$s = \sin\Omega t$$
$$\gamma = \dot{\gamma}_0 (t - t')$$

As was done in the example, we now wish to compare the viscosity calculated from $\underline{\underline{\tau}}$ written in the stationary frame. To calculate η we must be in a flow such that

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\gamma}_0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \tag{746}$$

which is true in the rotating frame when $t = 0$.

taking $t=0$ we obtain in the stationary frame:

$$\underline{\underline{\tau}}(0) = -\int_{-\infty}^{0} \frac{\eta_0}{\lambda^2} e^{\frac{(t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} dt'$$

$$\gamma = (t-t')\dot{\gamma}_0 = -t'\dot{\gamma}_0$$

which is independent of Ω and identical to $\underline{\underline{\tau}}(0)$ calculated from the rotating frame at $t=0$.

The Lodge eqn passes this test of invariance.