An Introduction to Fluid Mechanics

Faith A. Morrison
Professor of Chemical Engineering
Michigan Technological University

A caution about sign convention

In *Understanding Rheology* (Morrison, 2001) and *An Introduction to Fluid Mechanics* (Morrison, 2012) two different stress sign conventions are used. In the rheology text we follow Bird, Armstrong, and Hassager, *Dynamics of Polymeric Fluids* (Wiley, 1986) ($\underline{\underline{\Pi}} = -\underline{\underline{\tilde{\Pi}}}$, $\underline{\underline{\tau}} = -\underline{\underline{\tilde{\tau}}}$),), while in the fluids text we follow the usual engineering sign convention ($\underline{\underline{\tilde{\Pi}}}, \underline{\underline{\tilde{\tau}}}$). Any express that contains $\underline{\underline{\Pi}}$ or $\underline{\underline{\tau}}$ is affected.[§]

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$$

$$\underline{\underline{\Pi}} = p\underline{\underline{I}} + \underline{\underline{\tau}} \quad \text{(Bird et al.)}$$

$$\underline{\underline{\tilde{\tau}}} = +\mu\left(\nabla\underline{v} + \left(\nabla\underline{v}\right)^T\right)$$

$$\underline{\underline{\tau}} = -\mu\left(\nabla\underline{v} + \left(\nabla\underline{v}\right)^T\right) \quad \text{(Bird et al.)}$$

Force on surface with unit normal $\hat{n}$ and area S:

$$\underline{F} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS = \iint_S \left[\hat{n} \cdot (-\underline{\underline{\Pi}})\right]_{surface} dS$$

Torque on surface with unit normal $\hat{n}$ and area S:

$$\underline{T} = \iint_S \underline{R} \times \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS = \iint_S \underline{R} \times \left[\hat{n} \cdot (-\underline{\underline{\Pi}})\right]_{surface} dS$$

Sorry about that.

Equations for Inside Front Cover

Unit conversions summary: www.chem.mtu.edu/˜fmorriso/cm310/convert.pdf

Mechanical Energy Balance

$$\frac{\Delta p}{\rho} + \frac{\Delta \langle v \rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by\ fluid}}{m} \qquad \begin{cases} \alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1 \end{cases}$$

$$F_{friction} = \left[4f\frac{L}{D} + \sum_{fittings_i} n_i K_{f,i} \right] \frac{\langle v \rangle^2}{2}$$

Note error on inside front cover:

Fanning Friction Factor (pipe flow)

$$f = \frac{\mathcal{F}_{\text{drag}}}{\frac{1}{2}\rho\langle v \rangle^2 \cancel{\pi R^2}\ \mathbf{2\pi RL}} = \frac{\Delta p D}{2L\rho\langle v \rangle^2}$$

Drag Coefficient (sphere drop)

$$C_D = \frac{\mathcal{F}_{\text{drag}}}{\frac{1}{2}\rho v_\infty^2 \pi R^2} = \frac{4gD\left(\rho_{\text{body}} - \rho\right)}{3\rho v_\infty^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\mathbf{P}}{dt} + \iint_{CS} (\hat{n} \cdot \underline{v})\ \rho \underline{v}\ dS = \sum_{\substack{\text{on} \\ \text{CV}}} \underline{f}$$

Hydrostatic pressure

$$p_{bottom} = p_{top} + \rho g h$$

Hagen-Poiseuille equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Stokes-Einstein-Sutherland equation (steady, slow flow around a sphere)

$$\mathcal{F}_{\text{drag}} = 6\pi R\mu v_\infty$$

Macroscopic Momentum Balance on a CV

$$\frac{d\mathbf{P}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos\theta \langle v \rangle^2}{\beta}\ \hat{v} \right]\Bigg|_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n} \right]_{A_i} + \underline{R} + M_{CV}\underline{g} \qquad \begin{cases} \beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1 \end{cases}$$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla \cdot \underline{v} = 0$$

Total stress tensor $\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure $\mathcal{P} \equiv p + \rho g h$

Newtonian constitutive equation

$$\begin{aligned} \underline{\underline{\tilde{\tau}}} &= \mu \left(\nabla \underline{v} + (\nabla \underline{v})^T \right) \\ &= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123} \end{aligned}$$

Total molecular fluid force on a finite surface $\mathcal{S}$

$$\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right]_{\text{at surface}} dS$$

Stationary fluid $\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n}$

Moving fluid $\left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right] = -p\hat{n} + \hat{n} \cdot \underline{\underline{\tilde{\tau}}}$

Total fluid torque on a finite surface $\mathcal{S}$

$$\underline{\mathcal{T}} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\underline{\tilde{\Pi}}} \right) \right]_{\text{at surface}} dS$$

Total flow rate out through a finite surface $\mathcal{S}$

$$Q = \dot{V} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{v} \right]_{\text{at surface}} dS$$

Average velocity across a finite surface $\mathcal{S}$

$$\langle v \rangle = \frac{Q}{\mathcal{S}}$$

Coordinate system	surface differential dS
Cartesian (top, $\hat{n}=\hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n}=\hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n}=\hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n}=\hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n}=\hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n}=\hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta drd\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r\sin\theta\cos\phi$	$\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$
	$y = r\sin\theta\sin\phi$	$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$
	$z = r\cos\theta$	$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$
cylindrical	$x = r\cos\theta$	$\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$
	$y = r\sin\theta$	$\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$
	$z = z$	$\hat{e}_z = \hat{e}_z$

Divergence Theorem
$$\iint_{\mathcal{S}} \hat{n}\cdot\underline{F}\; dS = \iiint_{\mathcal{V}} \nabla\cdot\underline{F}\; dV$$

Stokes Theorem
$$\oint_{\mathcal{C}} \hat{t}\cdot\underline{F}\; dl = \iint_{\mathcal{S}} \hat{n}\cdot\left(\nabla\times\underline{F}\right)\; dS$$

Vector identities:

$$\begin{aligned}
\nabla\cdot\nabla\times\underline{F} &= 0 \quad \text{(Divergence of curl} = 0) \\
\nabla\times\nabla f &= 0 \quad \text{(Curl of gradient} = 0) \\
\nabla(fg) &= f\nabla g + g\nabla f \\
\underline{F}\cdot\nabla\underline{F} &= \frac{1}{2}\nabla\left(\underline{F}^2\right) - \underline{F}\times(\nabla\times\underline{F}) \\
\nabla\cdot(f\underline{F}) &= f\nabla\cdot\underline{F} + \underline{F}\cdot\nabla f \\
\nabla\times\nabla\times\underline{F} &= \nabla(\nabla\cdot\underline{F}) - \nabla^2\underline{F} \\
\nabla\cdot(\underline{F}\times\underline{G}) &= \underline{G}\cdot(\nabla\times\underline{F}) - \underline{F}\cdot(\nabla\times\underline{G})
\end{aligned}$$