Generalized Linear-Viscoelastic Model: (strain version)

infinitesimal strain tensor

$$\underline{\underline{\tau}} = + \int_{-\infty}^{t} M(t-t')\, \underline{\underline{\gamma}}(t,t')\, dt'$$

$$M(t-t') \equiv \frac{\partial G(t-t')}{\partial t'}$$

memory function

It is the use of the infinitesimal strain tensor as the strain measure that causes the frame-variance in the GLVE model.

What's wrong w/ using $\underline{\underline{\gamma}}(t,t')$ as our strain measure?

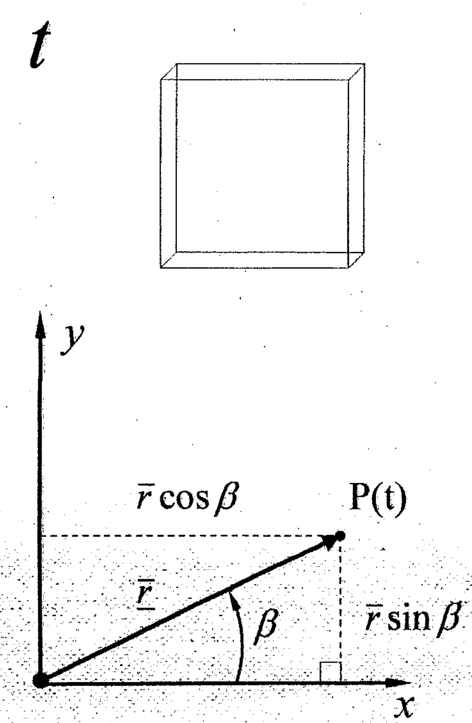
infinitesimal strain tensor

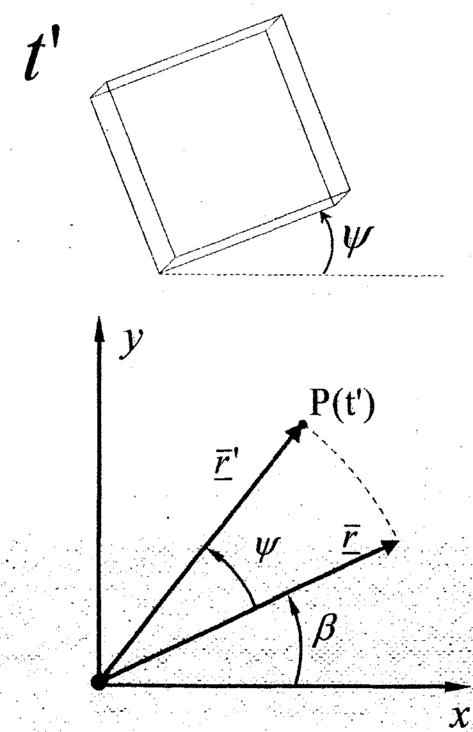
Defn $\underline{\underline{\gamma}}(t,t') = \nabla \underline{u}(t,t') + [\nabla \underline{u}(t,t')]^T$

$$\nabla u(t,t') = \underbrace{\underline{r}(t') - \underline{r}(t)}$$

difference between the position vector for a moving point at two times

No stress is generated when a fluid is rotated;
<u>what does the GLVE predict?</u>





$$\underline{r} = \underline{r}(\overset{\text{reference time}}{t}) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz} = \bar{\underline{r}} + z\,\hat{e}_z$$

$$\underline{r}' = \underline{r}(t') = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{xyz} = \bar{\underline{r}}' + z'\,\hat{e}_z$$

note: $z = z'$
$|\bar{\underline{r}}| = |\bar{\underline{r}}'|$

Looking for $\underline{u} = \underline{r}' - \underline{r} = \bar{\underline{r}}' - \bar{\underline{r}}$

$$\bar{\underline{r}}' = \begin{pmatrix} \bar{r}\cos(\psi + \beta) \\ \bar{r}\sin(\psi + \beta) \\ 0 \end{pmatrix}_{xyz}$$

$x = \bar{r}\cos\beta$
$y = \bar{r}\sin\beta$

$$\bar{\underline{r}}' = \begin{pmatrix} \bar{r}\left[\cos\psi\cos\beta - \sin\psi\sin\beta\right] \\ \bar{r}\left[\sin\psi\cos\beta + \cos\psi\sin\beta\right] \\ 0 \end{pmatrix}_{xyz}$$

$$\bar{\underline{r}}' = \begin{pmatrix} x\cos\psi - y\sin\psi \\ x\sin\psi + y\cos\psi \\ 0 \end{pmatrix}_{xyz} \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz}$$

$$\bar{\underline{r}} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}_{xyz}$$

$$\underline{u} = \underline{r}' - \underline{r} = \bar{\underline{r}}' - \underline{r}$$

$$\underline{u} = \begin{pmatrix} (\cos\psi - 1)x - y\sin\psi \\ x\sin\psi + y(\cos\psi - 1) \\ 0 \end{pmatrix}_{xyz}$$

$$\nabla\underline{u} + (\nabla\underline{u})^T = \underline{\underline{\gamma}}$$

carrying out this calculation we obtain:

$$\underline{\underline{\gamma}} = \begin{pmatrix} 2(\cos\psi - 1) & 0 & 0 \\ 0 & 2(\cos\psi - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz}$$

GLVE Prediction for Rigid-Body Rotation around the z-axis

$$\boxed{\underline{\underline{\tau}} = +\int_{-\infty}^{t} M(t-t') \begin{pmatrix} 2(\cos\psi - 1) & 0 & 0 \\ 0 & 2(\cos\psi - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz} dt'}$$

Why does GLVE make this erroneous prediction?

$$\underline{\underline{\gamma}}(t,t') = \nabla \underline{u}(t,t') + \left[\nabla \underline{u}(t,t')\right]^T$$

$$\underline{u}(t,t') = \underline{r}(t') - \underline{r}(t)$$

Because this vector, while accounting for deformation, ***also accounts for changes in orientation***.

34-11

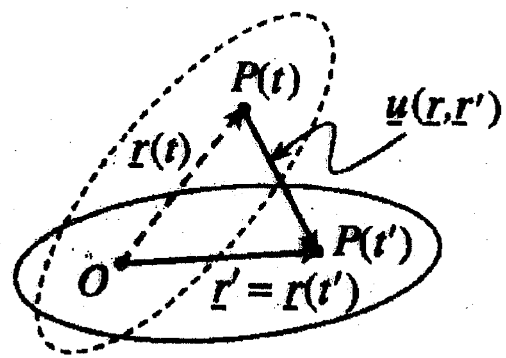
$$\underline{\underline{\gamma}}(t,t') = \nabla \underline{u}(t,t') + \left[\nabla \underline{u}(t,t')\right]^T$$

$$\underline{u}(t,t') = \underline{r}(t') - \underline{r}(t)$$

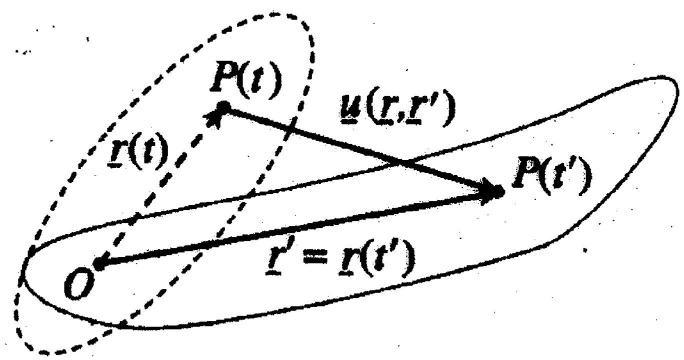
Accounts for changes in shape and orientation.

$\underline{u}(\underline{r},\underline{r}') = \underline{r}' - \underline{r}$

Origin O
fixed in space



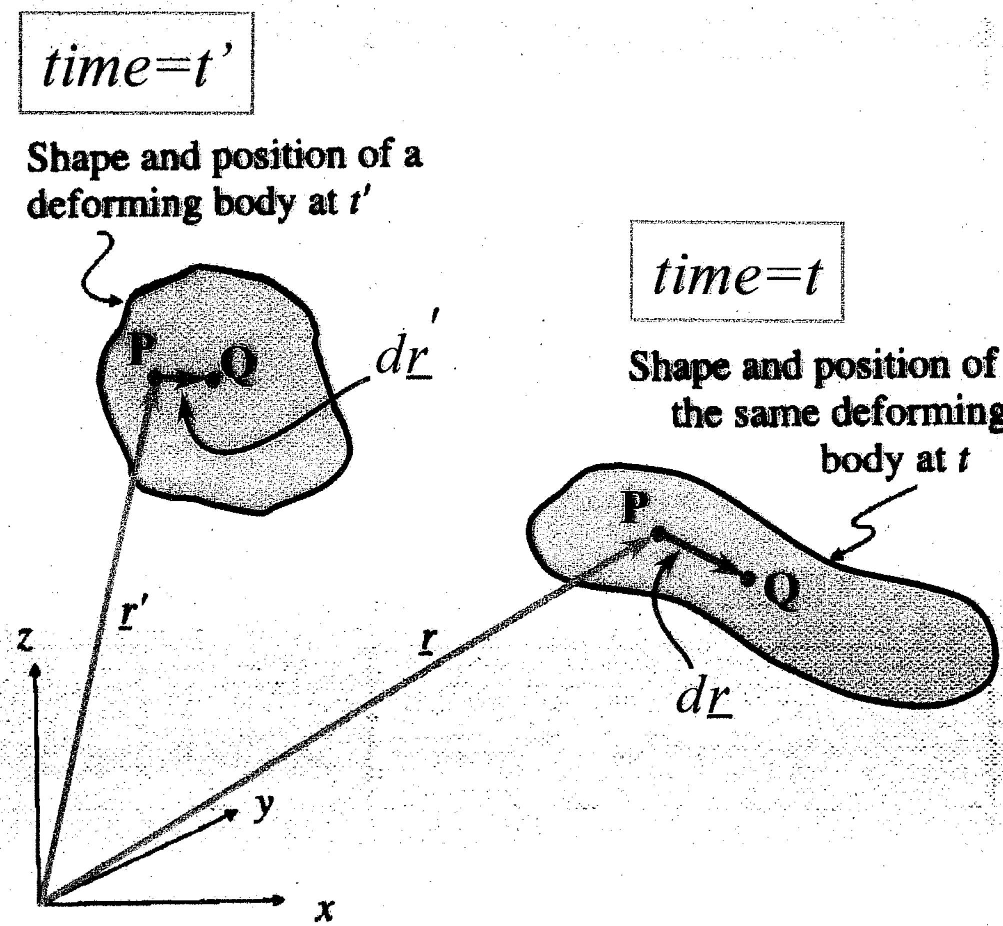
Orientation changes
($\underline{r}$ changes direction)
Shape does not change
(length of $\underline{r}$ does not change)



Orientation changes
Shape changes

We desire a strain tensor that accurately captures large-strain deformation without being affected by rigid-body rotation.

Consider:



fixed coordinate system (xyz)

⇒ consider the position of a particle at time t'

To identify which particle I'm talking about, I'll use its position at t

$\underline{r}'(t'; \underline{r})$ = position at t' of the particle that at t was at position $\underline{r}$

$$\underline{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{\hat{x}\hat{y}\hat{z}} \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz}$$

$$d\underline{r}' = \begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix}_{xyz}$$

write using chain rule

$\underline{r}' = \underline{r}'(t'; \underline{r})$

$$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy + \frac{\partial x'}{\partial z} dz$$

$$dy' = \frac{\partial y'}{\partial x} dx + \frac{\partial y'}{\partial y} dy + \frac{\partial y'}{\partial z} dz$$

$$dz' = \frac{\partial z'}{\partial x} dx + \frac{\partial z'}{\partial y} dy + \frac{\partial z'}{\partial z} dz$$

NOTE - this can be written as,

$$dx' = \frac{\partial x'}{\partial \underline{r}} \cdot d\underline{r}$$

$$= \frac{\partial x'}{\partial x_i} \underbrace{\hat{e}_i \cdot dx_p \hat{e}_p}_{\delta_{ip}} = \frac{\partial x'}{\partial x_p} dx_p$$

$$dy' = \frac{\partial y'}{\partial \underline{r}} \cdot d\underline{r}$$

$$dz' = \frac{\partial z'}{\partial \underline{r}} \cdot d\underline{r}$$

OR $\boxed{d\underline{r}' = \frac{\partial \underline{r}'}{\partial \underline{r}} \cdot d\underline{r}}$ $\equiv \underline{\underline{F}}$

$$\boxed{d\underline{r}' = \underline{\underline{F}} \cdot d\underline{r}}$$

Let $\underline{\underline{F}}^{-1}$ be the inverse of $\underline{\underline{F}}$

$$d\underline{r}' = \underline{\underline{F}} \cdot d\underline{r}$$

$$\underline{\underline{F}}^{-1} \cdot \underline{\underline{F}} = \underline{\underline{I}}$$

$$\underline{\underline{F}} \cdot \underline{\underline{F}}^{-1} = \underline{\underline{I}}$$

$$\underline{\underline{F}}^{-1} \cdot d\underline{r}' = \underbrace{\underline{\underline{F}}^{-1} \cdot \underline{\underline{F}}}_{\underline{\underline{I}}} \cdot d\underline{r}$$

$$\boxed{\underline{\underline{F}}^{-1} \cdot d\underline{r}' = d\underline{r}}$$

inverse
deformation
gradient tensor

compare: $\underline{\underline{F}} \cdot d\underline{r} = d\underline{r}'$

Deformation-gradient tensor

$$\underline{\underline{F}}(t,t') \equiv \frac{\partial \underline{r}'}{\partial \underline{r}} = \frac{\partial r_i'}{\partial r_p}\hat{e}_p\hat{e}_i = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz}$$

Inverse deformation-gradient tensor

$$\underline{\underline{F}}^{-1}(t',t) \equiv \frac{\partial \underline{r}}{\partial \underline{r}'} = \frac{\partial r_m}{\partial r_j'}\hat{e}_j\hat{e}_m = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz}$$

25 6

EXAMPLE: What is the inverse-deformation gradient tensor in steady shear flow?

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 y \\ 0 \\ 0 \end{pmatrix}_{xyz} \qquad \underline{r} = \begin{pmatrix} x' + (t - t')\dot{\gamma}_0 y' \\ y' \\ z' \end{pmatrix}_{xyz}$$

35 7

Calculate $\underline{\underline{F}}$, $\underline{\underline{F}}^{-1}$ for shear

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz} = \begin{pmatrix} x' + \overbrace{(t-t')}^{\text{time dif.}}\,\overbrace{\dot{\gamma}_0}^{\text{velocity}}\, y' \\ y' \\ z' \end{pmatrix}_{xyz}$$

$$\underline{\underline{F}} = \frac{\partial \underline{r}'}{\partial \underline{r}} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz}$$

Need to invert this

$$x' + (t-t')\,\dot{\gamma}_0\, y' = x$$
$$y' = y$$
$$z' = z$$

SOLVE FOR x', y', z'

tensor	shear in 1-direction with gradient in 2-direction	uniaxial elongation in 3-direction	ccw rotation around $\hat{e}_3$
$\underline{\underline{F}}(t,t')$	$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{-\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{F}}^{-1}(t',t)$	$\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{C}}(t,t')$	$\begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon} & 0 & 0 \\ 0 & e^{\epsilon} & 0 \\ 0 & 0 & e^{-2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{C}}^{-1}(t',t)$	$\begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon} & 0 & 0 \\ 0 & e^{-\epsilon} & 0 \\ 0 & 0 & e^{2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{\gamma}}^{[0]}(t,t')$	$\begin{pmatrix} 0 & -\gamma & 0 \\ -\gamma & \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon}-1 & 0 & 0 \\ 0 & e^{\epsilon}-1 & 0 \\ 0 & 0 & e^{-2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$
$\underline{\underline{\gamma}}_{[0]}(t,t')$	$\begin{pmatrix} -\gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon}-1 & 0 & 0 \\ 0 & e^{-\epsilon}-1 & 0 \\ 0 & 0 & e^{2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$

Table 9.3: Strain tensors for shear and extension in Cartesian coordinates. For shear flows $\gamma = \gamma(t',t) = \int_{t'}^{t} \varsigma(t'')\, dt'' = \int_{t'}^{t} \dot{\gamma}_{21}(t'')\, dt''$ and for elongational flows $\epsilon = \epsilon(t',t) = \int_{t'}^{t} \dot{\epsilon}(t'')\, dt''$. The angle ψ is the angle from $\underline{r}(t) = \underline{r}$ to $\underline{r}(t') = \underline{r}'$ in counter-clockwise (ccw) rotation around the $\hat{e}_3$-axis.

/e desire a strain tensor that accurately captures large-strain
eformation without being affected by rigid-body rotation.

$$\left.\begin{array}{c} \nabla \underline{u} \\ \underline{\underline{\gamma}} \\ \underline{\underline{F}} \\ \underline{\underline{F}}^{-1} \end{array}\right\}$$

All these strain measures include both deformation and orientation

We can separate the deformation and orientation information in $\underline{\underline{F}}$ and $\underline{\underline{F}}^{-1}$ using a technique called *polar decomposition*.

Solve for x', y', z' as a function of x, y, z

$$\boxed{\begin{aligned} x' &= x - (t-t')\,\dot{\gamma}_0\, y \\ y' &= y \\ z' &= z \end{aligned}}$$

Now, carry out derivatives in $\underline{\underline{F}}$

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -(t-t')\dot{\gamma}_0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

let $\gamma = \int_{t'}^{t} \dot{\gamma}_0 \, dt'' = \dot{\gamma}_0 (t-t')$

$$\underline{\underline{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

Polar Decomposition Theorem

Any tensor for which an inverse exists has two unique decompositions:

$$\boxed{\begin{aligned}\underline{\underline{A}} &= \underline{\underline{R}} \cdot \underline{\underline{U}} \\ &= \underline{\underline{V}} \cdot \underline{\underline{R}}\end{aligned}}$$

Pure rotation ten:

$$\underline{\underline{U}} = \left(\underline{\underline{A}}^T \cdot \underline{\underline{A}}\right)^{\frac{1}{2}}$$

$$\underline{\underline{V}} = \left(\underline{\underline{A}} \cdot \underline{\underline{A}}^T\right)^{\frac{1}{2}}$$

$$\underline{\underline{R}} = \underline{\underline{A}} \cdot \left(\underline{\underline{A}}^T \cdot \underline{\underline{A}}\right)^{-\frac{1}{2}} = \underline{\underline{A}} \cdot \underline{\underline{U}}^{-1}$$

$$\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$$

Orthogonal tensor

$\underline{\underline{U}}, \underline{\underline{V}}$

Symmetric, nonsingular tensors

36 - 2 9.5

$$\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$$

consider an arbitrary vector $\underline{v}$.

$$\underline{\underline{R}} \cdot \underline{v} = \underline{w}$$

NOTE:

$$\underline{w} = \underbrace{(\underline{\underline{R}} \cdot \underline{v}) = \underline{v} \cdot \underline{\underline{R}}^T}_{\text{verify w/ Einstein notation}}$$

calculate $|\underline{w}|$

$$\sqrt{\underline{w} \cdot \underline{w}} = \left(\underbrace{\underline{\underline{R}} \cdot \underline{v}} \cdot \underline{\underline{R}} \cdot \underline{v} \right)^{\frac{1}{2}}$$

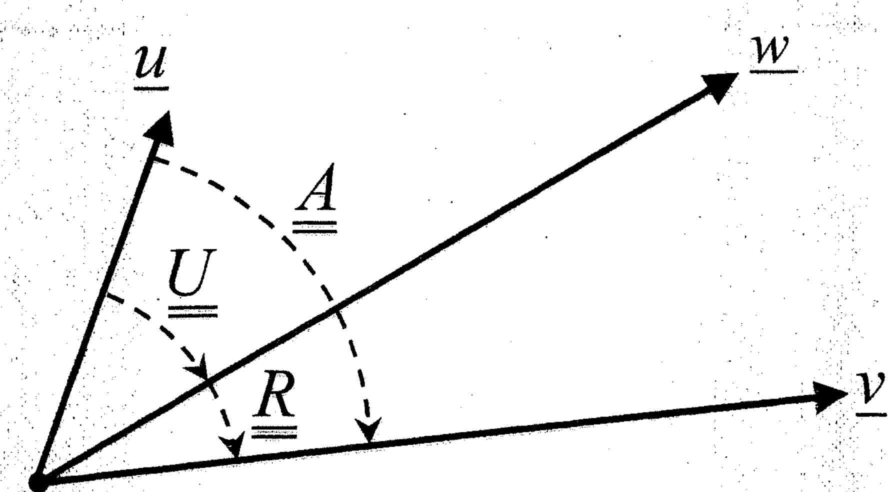
$$\left(\underline{v} \cdot \underbrace{\underline{\underline{R}}^T \cdot \underline{\underline{R}}}_{\underline{\underline{I}}} \cdot \underline{v} \right)^{\frac{1}{2}}$$

$$(\underline{v} \cdot \underline{v})^{\frac{1}{2}} = |\underline{w}|$$

$$= |\underline{v}|$$

KAMPLE: Calculate the right stretch tensor and rotation nsor for a given tensor. Calculate the angle through which $\underline{\underline{R}}$ tates the vector $\underline{u}$.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 2 \\ 2 & 0 & 0 \end{pmatrix}_{xyz} \qquad \underline{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{xyz}$$



10

We have partially isolated the effect of rotation through polar decomposition.

rotation tensor

left stretch tensor

$$\underline{\underline{A}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

right stretch tensor

original (strain) tensor

We can further isolate stretch from rotation by considering the *eigenvectors* of $\underline{\underline{U}}$ and $\underline{\underline{V}}$.