C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{w} = \frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r}\frac{\partial w_z}{\partial \theta} - \frac{\partial w_\theta}{\partial z} \\ \frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r w_\theta)}{\partial r} - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

$$\nabla \underline{w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{\partial w_\theta}{\partial r} & \frac{\partial w_z}{\partial r} \\ \frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} & \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} & \frac{1}{r}\frac{\partial w_z}{\partial \theta} \\ \frac{\partial w_r}{\partial z} & \frac{\partial w_\theta}{\partial z} & \frac{\partial w_z}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_r)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_r}{\partial \theta^2} + \frac{\partial^2 w_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial w_\theta}{\partial \theta} \\ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_\theta)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_\theta}{\partial \theta^2} + \frac{\partial^2 w_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial w_r}{\partial \theta} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_z}{\partial \theta^2} + \frac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z}$$

F. A. Morrison, Understanding Rheology (Oxford, 2001)

$$\nabla \cdot \underline{\underline{A}} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(rA_{rr}) + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\partial A_{zr}}{\partial z} - \frac{A_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_{r\theta}) + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{z\theta}}{\partial z} + \frac{A_{\theta r} - A_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_{rz}) + \frac{1}{r}\frac{\partial A_{\theta z}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix}_{r\theta z} \quad \text{(C.3-10)}$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r\left(\frac{\partial w_r}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + u_z\left(\frac{\partial w_r}{\partial z}\right) \\ u_r\left(\frac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + u_z\left(\frac{\partial w_\theta}{\partial z}\right) \\ u_r\left(\frac{\partial w_z}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + u_z\left(\frac{\partial w_z}{\partial z}\right) \end{pmatrix}_{r\theta z} \quad \text{(C.3-11)}$$

TABLE C.4
Differential Operations in the Spherical Coordinate System r, θ, ϕ

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_\phi \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-1)}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi} \quad \text{(C.4-2)}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial a}{\partial \phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-3)}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial a}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial a}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 a}{\partial \phi^2} \quad \text{(C.4-4)}$$

$$\nabla \cdot \underline{w} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} \quad \text{(C.4-5)}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\phi \sin\theta) - \frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r w_\phi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r w_\theta) - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-6)}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-7)}$$

continued

$$\nabla \underline{w} = \begin{pmatrix} \dfrac{\partial w_r}{\partial r} & \dfrac{\partial w_\theta}{\partial r} & \dfrac{\partial w_\phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} - \dfrac{w_\theta}{r} & \dfrac{1}{r}\dfrac{\partial w_\theta}{\partial \theta} + \dfrac{w_r}{r} & \dfrac{1}{r}\dfrac{\partial w_\phi}{\partial \theta} \\ \dfrac{1}{r \sin\theta}\dfrac{\partial w_r}{\partial \phi} - \dfrac{w_\phi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial w_\theta}{\partial \phi} - \dfrac{w_\phi}{r}\cot\theta & \dfrac{1}{r\sin\theta}\dfrac{\partial w_\phi}{\partial \phi} + \dfrac{w_r}{r} + \dfrac{w_\theta}{r}\cot\theta \end{pmatrix}_{r\theta\phi} \tag{C.4-8}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \left\{ \dfrac{\partial}{\partial r}\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 w_r)\right] + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial w_r}{\partial \theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_r}{\partial \phi^2} \right. \\ \left. - \dfrac{2}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}(w_\theta \sin\theta) - \dfrac{2}{r^2\sin\theta}\dfrac{\partial w_\phi}{\partial \phi} \right\} \\ \left\{ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial w_\theta}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(w_\theta \sin\theta)\right] + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_\theta}{\partial\phi^2} \right. \\ \left. + \dfrac{2}{r^2}\dfrac{\partial w_r}{\partial \theta} - \dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial w_\phi}{\partial\phi} \right\} \\ \left\{ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial w_\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(w_\phi \sin\theta)\right] + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_\phi}{\partial\phi^2} \right. \\ \left. + \dfrac{2}{r^2\sin\theta}\dfrac{\partial w_r}{\partial \phi} + \dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial w_\theta}{\partial\phi} \right) \end{pmatrix}_{r\theta\phi} \tag{C.4-9}$$

$$\nabla \cdot \underline{\underline{A}} = \begin{pmatrix} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_{rr}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta r}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi r}}{\partial \phi} - \dfrac{A_{\theta\theta} + A_{\phi\phi}}{r} \\ \dfrac{1}{r^3}\dfrac{\partial}{\partial r}(r^3 A_{r\theta}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta\theta}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi\theta}}{\partial\phi} + \dfrac{(A_{\theta r} - A_{r\theta}) - A_{\phi\phi}\cot\theta}{r} \\ \dfrac{1}{r^3}\dfrac{\partial}{\partial r}(r^3 A_{r\phi}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta\phi}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi\phi}}{\partial\phi} + \dfrac{(A_{\phi r} - A_{r\phi}) + A_{\phi\theta}\cot\theta}{r} \end{pmatrix}_{r\theta\phi} \tag{C.4-10}$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r\left(\dfrac{\partial w_r}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_r}{\partial\theta} - \dfrac{w_\theta}{r}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_r}{\partial\phi} - \dfrac{w_\phi}{r}\right) \\ u_r\left(\dfrac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_\theta}{\partial\theta} + \dfrac{w_r}{r}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_\theta}{\partial\phi} - \dfrac{w_\phi}{r}\cot\theta\right) \\ u_r\left(\dfrac{\partial w_\phi}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_\phi}{\partial\theta}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_\phi}{\partial\phi} + \dfrac{w_r}{r} + \dfrac{w_\theta}{r}\cot\theta\right) \end{pmatrix}_{r\theta\phi} \tag{C.4-11}$$

tensor	shear in 1-direction with gradient in 2-direction	uniaxial elongation in 3-direction	ccw rotation around $\hat{e}_3$
$\underline{\underline{F}}(t,t')$	$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{-\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{F}}^{-1}(t',t)$	$\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{C}}(t,t')$	$\begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon} & 0 & 0 \\ 0 & e^{\epsilon} & 0 \\ 0 & 0 & e^{-2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{C}}^{-1}(t',t)$	$\begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon} & 0 & 0 \\ 0 & e^{-\epsilon} & 0 \\ 0 & 0 & e^{2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{\gamma}}^{[0]}(t,t')$	$\begin{pmatrix} 0 & -\gamma & 0 \\ -\gamma & \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon}-1 & 0 & 0 \\ 0 & e^{\epsilon}-1 & 0 \\ 0 & 0 & e^{-2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$
$\underline{\underline{\gamma}}_{[0]}(t,t')$	$\begin{pmatrix} -\gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon}-1 & 0 & 0 \\ 0 & e^{-\epsilon}-1 & 0 \\ 0 & 0 & e^{2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$

Table 9.3: Strain tensors for shear and extension in Cartesian coordinates. For shear flows $\gamma = \gamma(t',t) = \int_{t'}^{t} \dot{\varsigma}(t'')\, dt'' = \int_{t'}^{t} \dot{\gamma}_{21}(t'')\, dt''$ and for elongational flows $\epsilon = \epsilon(t',t) = \int_{t'}^{t} \dot{\epsilon}(t'')\, dt''$. The angle ψ is the angle from $\underline{r}(t) = \underline{r}$ to $\underline{r}(t') = \underline{r}'$ in counter-clockwise (ccw) rotation around the $\hat{e}_3$-axis.

reverse ↺

Errata: www.chem.mtu.edu/~fmorriso/UKerrata.html