Mathematics Review

1. Scalar – *a mathematical entity that has magnitude only*

e.g.: temperature T
speed *v*
time *t*
density ρ

– *scalars may be constant or may be variable*

Laws of Algebra for Scalars:

yes commutative $ab = ba$

yes associative $a(bc) = (ab)c$

yes distributive $a(b+c) = ab+ac$

2. Vector – *a mathematical entity that has magnitude and direction*

e.g.: force on a surface $\underline{f}$
velocity $\underline{v}$

– *vectors may be constant or may be variable*

Definitions

magnitude of a vector – a scalar associated with a vector

$$|\underline{v}| = v \qquad |\underline{f}| = f$$

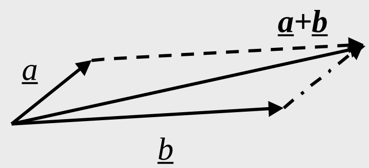
unit vector – a vector of unit length

$$\frac{\underline{v}}{|\underline{v}|} = \hat{\underline{v}}$$

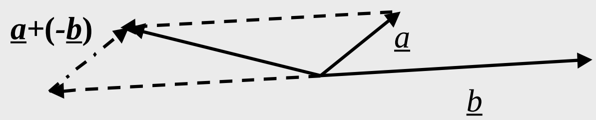
a unit vector in the direction of $\underline{v}$

Laws of Algebra for Vectors:

1. Addition



2. Subtraction



Laws of Algebra for Vectors (continued):

3. Multiplication by scalar $\alpha \underline{v}$

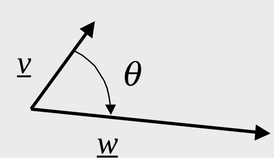
yes commutative $\alpha \underline{v} = \underline{v} \alpha$

yes associative $\alpha(\beta \underline{v}) = (\alpha\beta)\underline{v} = \alpha\beta\underline{v}$

yes distributive $\alpha(\underline{v} + \underline{w}) = \alpha\underline{v} + \alpha\underline{w}$

4. Multiplication of vector by vector
   4a. scalar (dot) (inner) product

$$\underline{v} \cdot \underline{w} = vw\cos\theta$$



Note: we can find magnitude with dot product

$$\underline{v} \cdot \underline{v} = vv\cos 0 = v^2$$

$$v = |\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$$

Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con't)

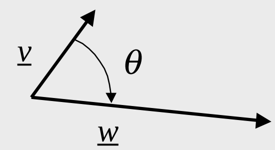
yes commutative $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$

NO associative ~~$\underline{v} \cdot \underline{w} \cdot \underline{z}$~~ no such operation

yes distributive $\underline{z} \cdot (\underline{v} + \underline{w}) = \underline{z} \cdot \underline{v} + \underline{z} \cdot \underline{w}$

4b. vector (cross) (outer) product

$$\underline{v} \times \underline{w} = vw \sin\theta \, \hat{e}$$



$\hat{e}$ is a unit vector perpendicular to both $\underline{v}$ and $\underline{w}$ following the right-hand rule

Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con't)

NO commutative $\underline{v} \times \underline{w} \neq \underline{w} \times \underline{v}$

yes associative $\underline{v} \times \underline{w} \times \underline{z} = (\underline{v} \times \underline{w}) \times \underline{z} = \underline{v} \times (\underline{w} \times \underline{z})$

yes distributive $\underline{z} \times (\underline{v} + \underline{w}) = (\underline{z} \times \underline{v}) + (\underline{z} \times \underline{w})$

Coordinate Systems

•Allow us to make actual calculations with vectors

Rule: any three vectors that are *non-zero* and *linearly independent* (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if α, β, and γ can be found such that:

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = \underline{0}$$
$$for \quad \alpha, \beta, \gamma \neq 0$$

If α, β, and γ are found to be zero, the vectors are linearly independent.

How can we do actual calculations with vectors?

Rule: *any* vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

coefficient of $\underline{a}$ in the $\hat{e}_y$ direction

any vector

$$\underline{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{xyz}$$

$$= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$= \sum_{j=1}^{3} a_j \hat{e}_j$$

Trial calculation: dot product of two vectors

$$
\begin{aligned}
\underline{a}\cdot\underline{b} &= \left(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3\right)\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right)\\
&= a_1\hat{e}_1\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) +\\
&\quad a_2\hat{e}_2\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) +\\
&\quad a_3\hat{e}_3\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right)\\
&= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 +\\
&\quad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 +\\
&\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3
\end{aligned}
$$

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

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$$
\begin{aligned}
&\hat{e}_1\cdot\hat{e}_1 = 1\\
&\hat{e}_1\cdot\hat{e}_2 = 0\\
&\hat{e}_1\cdot\hat{e}_3 = 0\\
&\cdots
\end{aligned}
$$

$$
\begin{aligned}
\underline{a}\cdot\underline{b} = &\; a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 +\\
&\; a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 +\\
&\; a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3\\
= &\; a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}
$$

We can generalize this operation with a technique called Einstein notation.

Einstein Notation

a system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \sum_{j=1}^{3} a_j\hat{e}_j$$
$$= a_j\hat{e}_j = a_m\hat{e}_m$$

- the initial choice of subscript letter is *arbitrary*
- the presence of a pair of like subscripts implies a missing summation sign

Mathematics Review Polymer Rheology

Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$
$$\hat{e}_1 \cdot \hat{e}_2 = 0$$
$$\hat{e}_1 \cdot \hat{e}_3 = 0$$
$$\cdots$$

$$\underbrace{\hat{e}_i \cdot \hat{e}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases}}_{\text{Kronecker delta}}$$

Einstein Notation (con't)

To carry out a dot product of two arbitrary vectors . . .

Detailed Notation	Einstein Notation
$\underline{a}\cdot\underline{b}=(a_1\hat{e}_1+a_2\hat{e}_2+a_3\hat{e}_3)\cdot(b_1\hat{e}_1+b_2\hat{e}_2+b_3\hat{e}_3)$ $=a_1\hat{e}_1\cdot b_1\hat{e}_1+a_1\hat{e}_1\cdot b_2\hat{e}_2+a_1\hat{e}_1\cdot b_3\hat{e}_3+$ $a_2\hat{e}_2\cdot b_1\hat{e}_1+a_2\hat{e}_2\cdot b_2\hat{e}_2+a_2\hat{e}_2\cdot b_3\hat{e}_3+$ $a_3\hat{e}_3\cdot b_1\hat{e}_1+a_3\hat{e}_3\cdot b_2\hat{e}_2+a_3\hat{e}_3\cdot b_3\hat{e}_3$ $=a_1b_1+a_2b_2+a_3b_3$	$\underline{a}\cdot\underline{b}=a_j\hat{e}_j\cdot b_m\hat{e}_m$ $=a_j\delta_{jm}b_m$ $=a_jb_j$