Constitutive Equation – **an accounting for <u>all</u> stresses, all flows**

stress <u>tensor</u>

Newtonian fluids:
(all flows)

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$$

Rate-of-deformation <u>tensor</u>

In general:

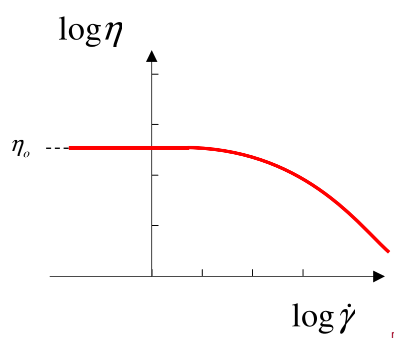
$$\underline{\underline{\tau}} = -f\left(\underline{\underline{\dot{\gamma}}}\right)$$

In the general case, f needs to be a non-linear function (in time and position)

What should we choose for the function *f*?

Non-Newtonian, Inelastic Fluids

First, we concentrate on the observation that **shear viscosity depends on shear rate**.



$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

Non-Newtonian viscosity, η

$$\dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right|$$

shear rate

We will design a constitutive equation that predicts this behavior in shear flow

Newtonian Constitutive Equation

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}}$$

For Newton's experiment (shear flow): $\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = -\mu \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

We could make this equation give the right answer (shear thinning) in steady shear flow if we substituted ***a function of shear rate*** for the constant viscosity.

Generalized Newtonian Fluid (GNF)
constitutive equation

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = -\eta(\dot{\gamma}) \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

SHEAR FLOW: $\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$, $\dot{\gamma} \equiv \left| \frac{\partial v_1}{\partial x_2} \right|$

GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

ALL FLOWS: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$, $\dot{\gamma} \equiv \left| \dot{\underline{\underline{\gamma}}} \right|$

Rate of Deformation $\dot{\gamma}$

Magnitude of the rate of deformation tensor

$$\dot{\gamma} = \left|\underline{\underline{\dot{\gamma}}}\right| \equiv \left|\sqrt{\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{ij}\dot{\gamma}_{ji}}\right|$$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} \dot{\gamma}_{11} & \dot{\gamma}_{12} & \dot{\gamma}_{13} \\ \dot{\gamma}_{21} & \dot{\gamma}_{22} & \dot{\gamma}_{23} \\ \dot{\gamma}_{31} & \dot{\gamma}_{32} & \dot{\gamma}_{33} \end{pmatrix}_{123} = \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

Constitutive Equation – an accounting for <u>all</u> stresses, all flows

$$\underline{\underline{\tau}} = -f\left(\underline{\underline{\dot{\gamma}}}\right)$$

A simple choice for *f*:

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

WAIT!

What is our justification for this (or any other) choice?

Well, none really. It just seems like it might work. If it does, we've lucked out!

This is called an **empirical model.**

Thus, after proposing such a model, we will have to check to see if its predictions match reality. If they do, we have made a good (lucky!) choice.

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

Model characteristics:

- Stress tensor is directly proportional to rate-of-deformation tensor
- There is no explicit time-dependence
- The function η is only a function of the magnitude of the rate-of-deformation tensor

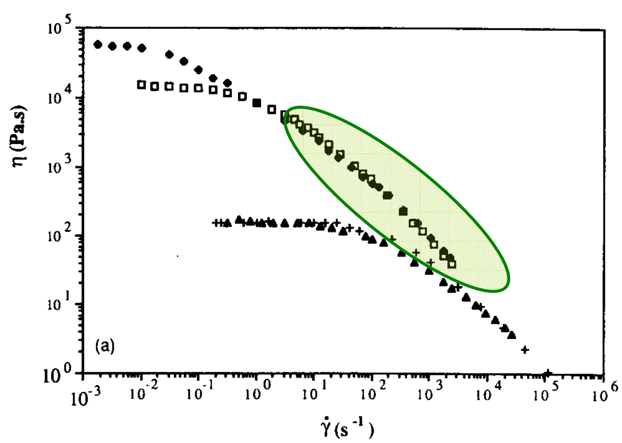
$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

What do we pick for $\eta(\dot{\gamma})$?

- Something that matches the data;
- Something simple, so that the calculations are easy

In processing, the high-shear-rate behavior is the most important.



+ linear 131 kg/mole
▲ branched 156 kg/mole
□ linear 418 kg/mol
◆ branched 428 kg/mol

Figure 6.3, p. 172 Piau et al., linear and branched PDMS

Power-law model for viscosity

$$\eta = m\dot{\gamma}^{n-1}$$

in shear flow $\dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right|$

$$\eta = m\left|\frac{dv_1}{dx_2}\right|^{n-1}$$

(in shear flow)

On a log-log plot, this would give a straight line:

$$\underbrace{\log\eta}_{Y} = \underbrace{\log m}_{B} + \underbrace{(n-1)}_{M}\underbrace{\log\left|\frac{dv_1}{dx_2}\right|}_{X}$$

$$Y = B + M X$$

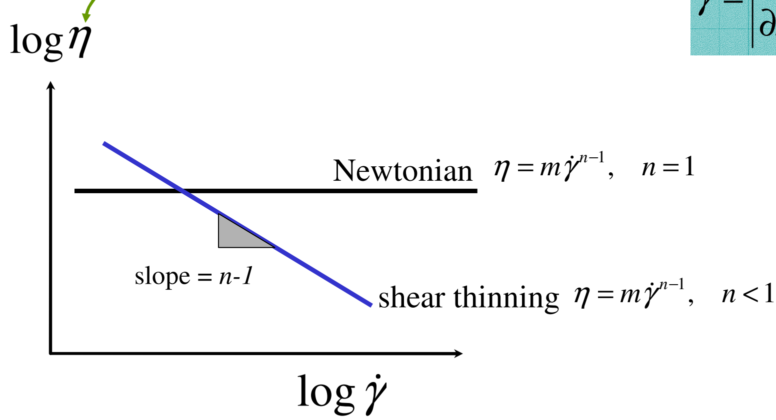
Power-law model for viscosity

steady shear flow

Non-Newtonian shear viscosity

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

$$\dot{\gamma} = \left|\frac{\partial v_1}{\partial x_2}\right|$$



Power-Law Generalized Newtonian Fluid (Ostwald-deWaele Model)

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta = -m\dot{\gamma}^{n-1}$$

m or *K* = consistency index (*m* = μ for Newtonian)
n = power-law index (*n* = 1 for Newtonian)

$$\dot{\gamma} \equiv \left|\underline{\underline{\dot{\gamma}}}\right|$$

(Usually 0.5≤ *n* ≤1)

Carreau-Yassuda GNF

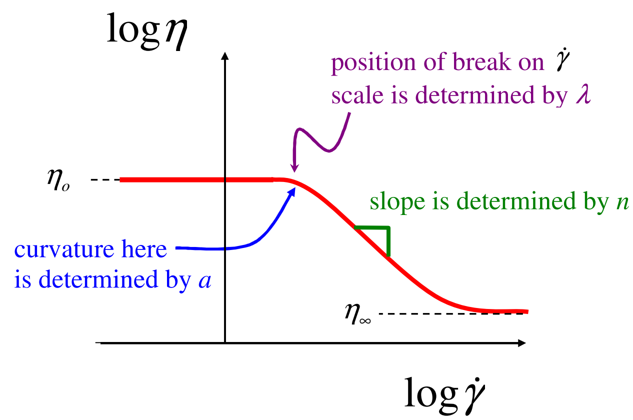
A model with 5 parameters

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + (\dot{\gamma}\lambda)^a\right]^{\frac{n-1}{a}}$$

- The viscosity function approaches the constant value of η_∞ as deformation rate get large
- The viscosity function approaches the constant value η_0 as deformation rate gets small
- λ is the time constant for the fluid
- *n* determines the slope of the power-law region

Carreau-Yassuda GNF



What about shear thickening?

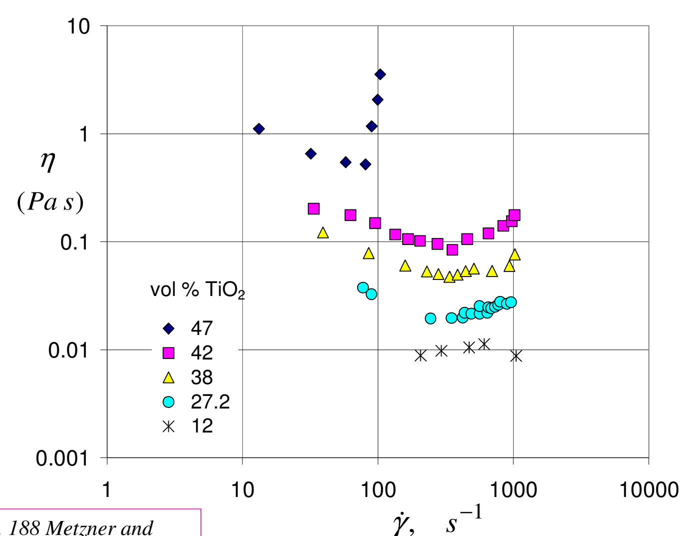
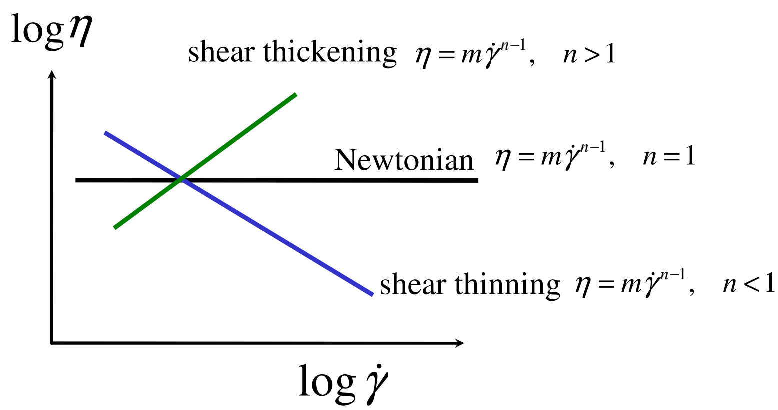


Figure 6.27, p. 188 Metzner and Whitlock; TiO_2/water suspensions

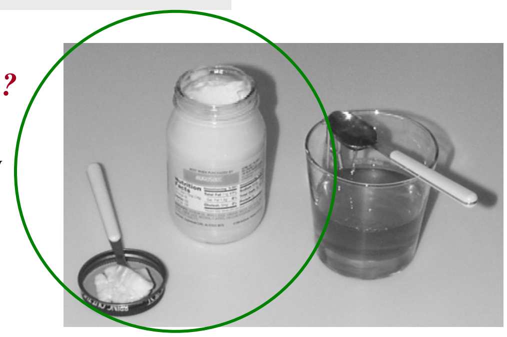
Power-Law GNF

steady shear flow



Other Inelastic Fluids

What about mayonnaise?



Mayonnaise and many other like fluids (paint, ketchup, most suspensions, asphalt) is able to sustain a **yield stress.**

Once the fluid begins to deform under an imposed stress, the viscosity may either be constant or may shear-thin. This type of steady shear viscosity behavior can be modeled with a GNF.

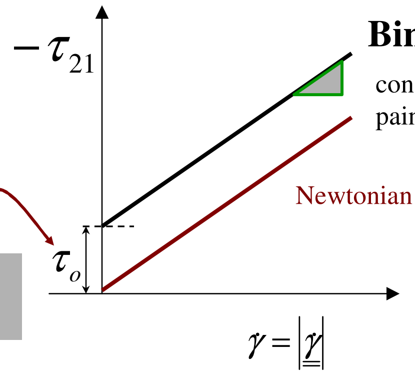
Non-Newtonian Fluids

For some fluids, no flow occurs when moderate stresses are applied.

Bingham plastic

constant slope = μ_o (mayo, paints, suspensions)

Yield stress

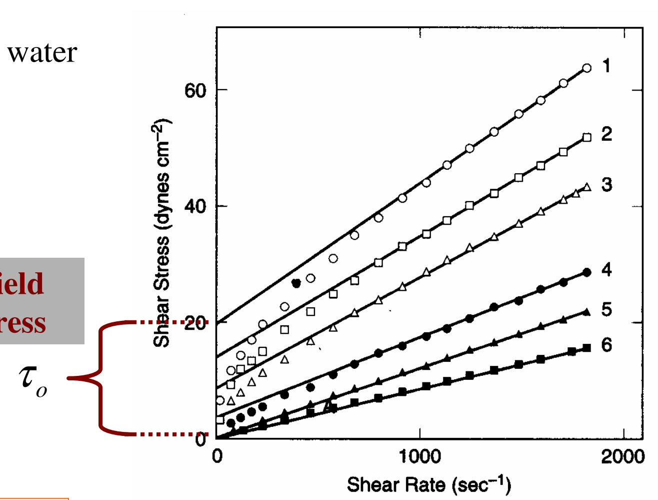


Non-Newtonian Fluids

Bingham plastic

PMMA in water

Yield stress



Friend and Hunter, 1971; dispersions of PMMA in water at various ζ-potentials; From Larson, p353.

Bingham Plastic
steady shear flow

Non-Newtonian viscosity, η

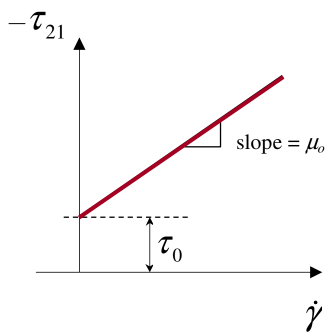
$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

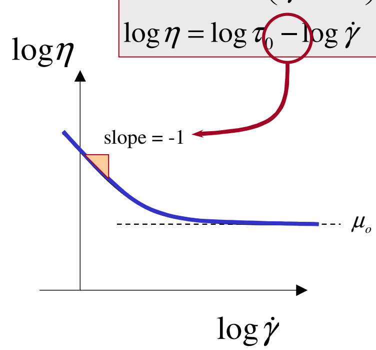
$$\eta = \frac{\tau_0}{\dot{\gamma}} + \mu_0$$

$$\lim_{\dot{\gamma}\to 0}(\eta) = \lim_{\dot{\gamma}\to 0}\left(\frac{\tau_0}{\dot{\gamma}} + \mu_0\right) = \frac{\tau_0}{\dot{\gamma}}$$

$$\log\eta = \log\tau_0 - \log\dot{\gamma}$$

$$-\tau_{21} \equiv \tau_0 + \mu_0\dot{\gamma}$$





Bingham GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

A model with 2 parameters

$$\eta(\dot{\gamma}) = \begin{cases} \infty & \left|\underline{\underline{\tau}}\right| \le \tau_0 \\ \mu_o + \dfrac{\tau_0}{\dot{\gamma}} & \left|\underline{\underline{\tau}}\right| > \tau_0 \end{cases}$$

μ_o = viscosity parameter
τ_y = yield stress

There is no flow until the shear stress exceeds a critical value τ_0 called the yield stress.

Other GNF viscosity models

See Carreau, DeKee, and Chhabra for complete discussion (*Rheology of Polymeric Systems*, Hanser, 1997)

Ellis Model $\eta = \dfrac{\eta_0}{1+\left|\dfrac{\tau}{\tau_0}\right|^{\alpha-1}}$ $\tau = \left|\underline{\underline{\tau}}\right|$

4-Parameter Carreau Model (same as CY with *a*=2)

Cross-Williamson Model (same as CY with *a*=1, $\eta_\infty = 0$)

DeKee Model $\eta = \eta_1 e^{-\lambda\dot{\gamma}} + \eta_2 e^{-0.1\lambda\dot{\gamma}} + \eta_\infty$

Casson Model $\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{\eta_0 \gamma}$ $\tau = \left|\underline{\underline{\tau}}\right|$

Herschel-Bulkley Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + m\dot{\gamma}^{n-1}$

DeKee-Turcotte Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + \eta_1 e^{-\lambda\dot{\gamma}}$

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Other GNF viscosity models

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Herschel-Bulkley Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + m\dot{\gamma}^{n-1}$

Yield stress plus power-law viscosity behavior

DeKee-Turcotte Model $\eta = \dfrac{\tau_0}{\dot{\gamma}} + \eta_1 e^{-\lambda\dot{\gamma}}$

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