Summary: *Generalized Newtonian Fluid Constitutive Equations*

PRO:

- A first constitutive equation
- Can match steady shearing data very well
- Simple to calculate with
- Found to predict pressure-drop/flow rate relationships well

CON:

- Fails to predict shear normal stresses
- Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
- Derived ad hoc from shear observations; unclear of validity in non-shear flows

We now look to address this failing of GNF models by seeking to incorporate *memory*.

Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f\left(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info}\right)$$

The stress expression:

- ***Must be of tensor order***
- ***Must be a tensor (independent of coordinate system)***
- ***Must be a symmetric tensor***
- ***Must make predictions that are independent of the observer***
- ***Should correctly predict observed flow/deformation behavior***

Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\dot{\gamma}}, II_{\dot{\gamma}}, III_{\dot{\gamma}}, \text{m}$$

Tensor invariants – scalars associated with a tensor that do not depend on coordinate system

The stress expression:

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Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\underline{\underline{A}} = tr\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace\underline{\underline{A}} = \sum_{p=1}^{3} A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = \sum_{p=1}^{3}\sum_{k=1}^{3} A_{pk} A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \sum_{p=1}^{3}\sum_{j=1}^{3}\sum_{h=1}^{3} A_{pj} A_{jh} A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

Fluids with Memory - Chapter 8

We seek a constitutive equation that includes memory effects.

$$\underline{\underline{\tau}}(t) = f(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material information})$$

calculates the stress at a particular time, t

2 equations so far:

$$\underline{\underline{\tau}}(t) = -\mu \underline{\underline{\dot{\gamma}}}(t)$$

$$\underline{\underline{\tau}}(t) = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}(t) \qquad \dot{\gamma} = \left|\underline{\underline{\dot{\gamma}}}\right|$$

So far, stress at t depends on rate-of-deformation **at *t only***

Current Constitutive Equations

Newtonian $\quad \underline{\underline{\tau}}(t) = -\mu \underline{\underline{\dot{\gamma}}}(t)$

Generalized Newtonian $\quad \underline{\underline{\tau}}(t) = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}(t) \qquad \dot{\gamma} = \left|\underline{\underline{\dot{\gamma}}}\right|$

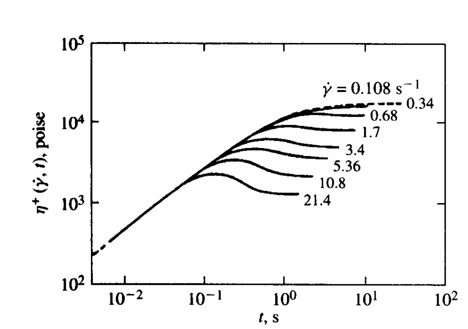
Neither can predict:

- Shear normal stresses - *this will be wrong so long as we use constitutive equations proportional to* $\underline{\underline{\dot{\gamma}}}$
- stress transients in shear (startup, cessation) - *this flaw seems to be related to omitting fluid memory*

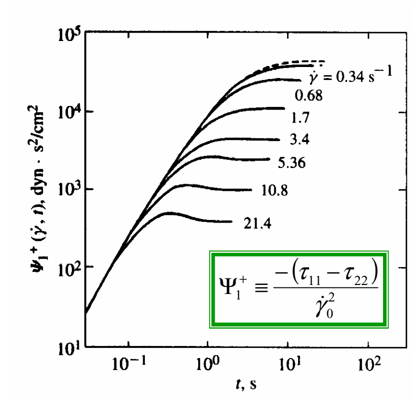
We will try to fix this now; we will address the first point when we discuss advanced constitutive equations

Startup of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$



$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

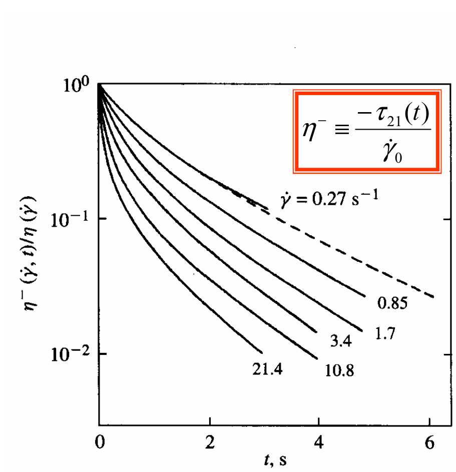


$$\Psi_1^+ \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

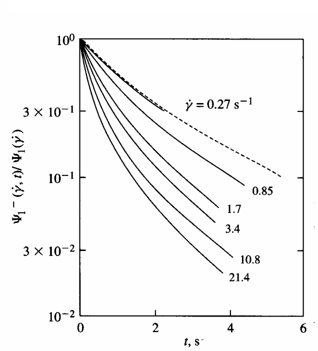
Figures 6.49, 6.50, p. 208
Menezes and Graessley, PB soln

Cessation of Steady Shearing

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$



$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$



$$\Psi_1^- \equiv \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$$

Figures 6.51, 6.52, p. 209 Menezes and Graessley, PB soln

How can we incorporate time-dependent effects?

First we explore a simple memory fluid.

Let's construct a new constitutive equation that remembers the stress at a time t_0 seconds ago

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\,\dot{\underline{\underline{\gamma}}}(t) - (0.8\tilde{\eta})\,\dot{\underline{\underline{\gamma}}}(t - t_0)$$

Newtonian contribution

contribution from fluid memory

This is the rate-of-deformation tensor t_0 seconds before time t

$\tilde{\eta}$ is a constant parameter of the model

What does this model predict?

Steady shear

$$\eta = ?$$
$$\Psi_1 = ?$$
$$\Psi_2 = ?$$

Shear start-up

$$\eta^+(t) = ?$$
$$\Psi_1^+(t) = ?$$
$$\Psi_2^+(t) = ?$$

Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \dot{\gamma}_0 = \text{constant}$$

Material Functions:

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0}$$

Viscosity

First normal-stress coefficient

$$\Psi_1 \equiv \frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-stress coefficient

$$\Psi_2 \equiv \frac{-(\tau_{22}-\tau_{33})}{\dot{\gamma}_0^2}$$

Start-up of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^+ \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

Shear stress growth function

First normal-stress growth function

$$\Psi_1^+ \equiv \frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0^2}$$

Second normal-stress growth function

$$\Psi_2^+ \equiv \frac{-(\tau_{22}-\tau_{33})}{\dot{\gamma}_0^2}$$

Cessation of Steady Shear Flow Material Functions

Kinematics:

$$\underline{v} \equiv \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$$

Material Functions:

$$\eta^- \equiv \frac{-\tau_{21}(t)}{\dot{\gamma}_0}$$

Shear stress decay function

First normal-stress decay function $\Psi_1^- \equiv \dfrac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}$

Second normal-stress decay function $\Psi_2^- \equiv \dfrac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}$

Predictions of the simple memory fluid

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\,\underline{\underline{\dot{\gamma}}}(t) - (0.8\tilde{\eta})\,\underline{\underline{\dot{\gamma}}}(t - t_0)$$

Steady shear

$$\eta = 1.8\tilde{\eta}$$

$$\Psi_1 = \Psi_2 = 0$$

The steady viscosity reflects contributions from what is currently happening and contributions from what happened t_0 seconds ago.

Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \leq t \leq t_0 \\ 1.8\tilde{\eta} & t \geq t_0 \end{cases}$$

$$\Psi_1^+(t) = \Psi_2^+(t) = 0$$

Predictions of the simple memory fluid

Shear start-up

$$\eta^+(t) = \begin{cases} 0 & t < 0 \\ \tilde{\eta} & 0 \le t \le t_0 \\ 1.8\tilde{\eta} & t \ge t_0 \end{cases}$$

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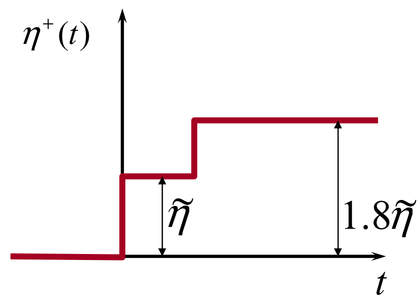
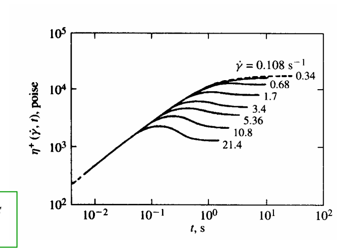


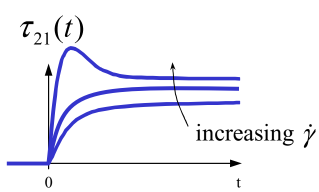


Figures 6.49, 6.50, p. 208 Menezes and Graessley, PB soln

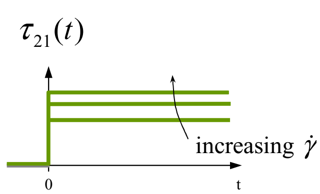
Predictions of the simple memory fluid

Shear start-up

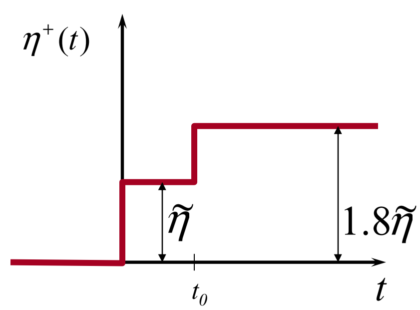
What the data show:



What the GNF models predict:

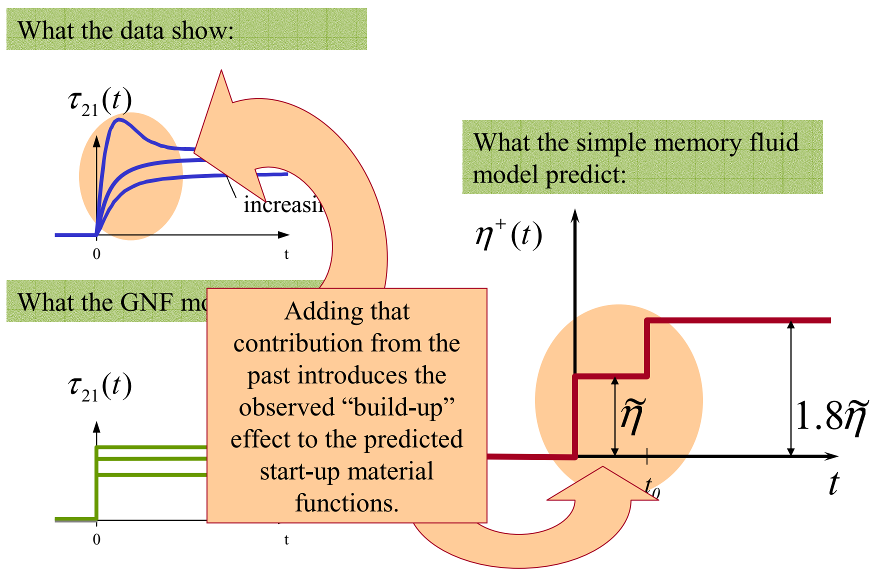


What the simple memory fluid model predict:



Predictions of the simple memory fluid

Shear start-up



We can make the stress rise smoother by adding more fading memory terms.

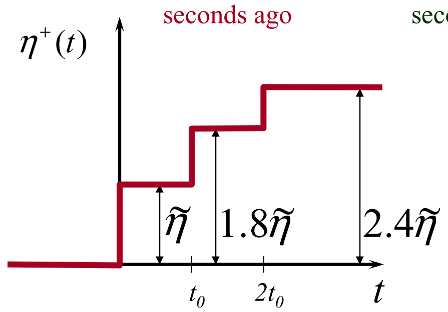
The memory is fading

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\,\underline{\underline{\gamma}}(t) - (0.8\tilde{\eta})\,\underline{\underline{\gamma}}(t-t_0) - (0.6\tilde{\eta})\,\underline{\underline{\gamma}}(t-2t_0)$$

Newtonian contribution

contribution from t_0 seconds ago

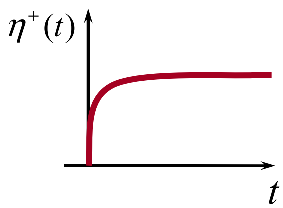
contribution from $2t_0$ seconds ago



The fit can be made to be perfectly smooth by using a sum of exponentially decaying terms as the weighting functions.

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\Big[\dot{\underline{\underline{\gamma}}}(t) + (0.37)\dot{\underline{\underline{\gamma}}}(t-t_0) + (0.14)\dot{\underline{\underline{\gamma}}}(t-2t_0) + (0.05)\dot{\underline{\underline{\gamma}}}(t-3t_0) + \ldots\Big]$$

$$= -\tilde{\eta} \sum_{p=0}^{\infty} e^{-p}\, \dot{\underline{\underline{\gamma}}}(t-pt_0)$$



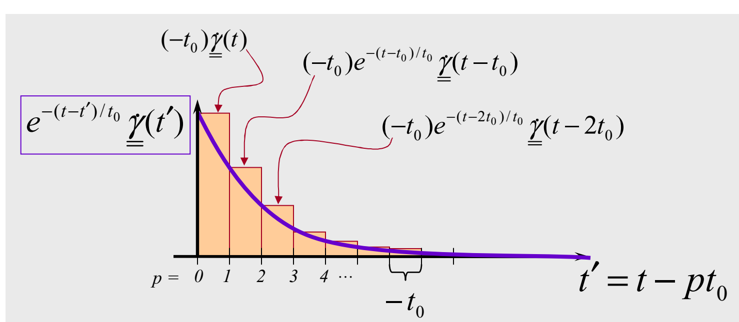
p	e^{-p}
0	1.00
1	0.37
2	0.14
3	0.05
4	0.02

This sum can be *approximated* by an integral.

$t'_p \equiv t - pt_0$

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\sum_{p=0}^{\infty} e^{-p}\, \underline{\underline{\gamma}}(t-pt_0)$$

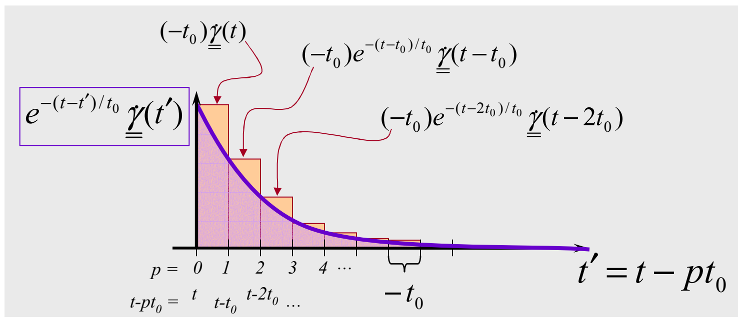
$$= -\tilde{\eta}\sum_{p=0}^{\infty} e^{-(t-t'_p)/t_0}\, \underline{\underline{\gamma}}(t'_p) = -\tilde{\eta}\,\frac{(area)}{(-t_0)}$$



This sum can be *approximated* by this integral.

$$\underline{\underline{\tau}}(t) = -\tilde{\eta}\,\frac{(area)}{(-t_0)} \approx \frac{\tilde{\eta}}{t_0}\int_t^{-\infty} e^{-(t-t')/t_0}\,\underline{\underline{\gamma}}(t')\,dt'$$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t}\left(\frac{\tilde{\eta}}{t_0}\right) e^{-(t-t')/t_0}\,\underline{\underline{\gamma}}(t')\,dt'$$



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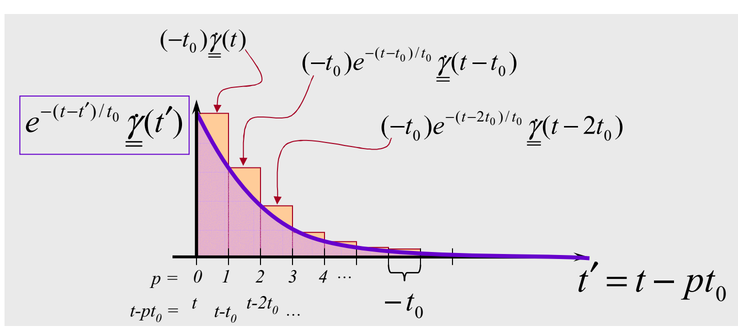
$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t}\left(\frac{\tilde{\eta}}{t_0}\right) e^{-(t-t')/t_0}\,\underline{\underline{\gamma}}(t')\,dt'$$

(Note: this is an underestimate of our previous decaying function, but the choice of function is arbitrary)



$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left(\frac{\tilde{\eta}}{t_0}\right) e^{-(t-t')/t_0}\, \underline{\underline{\dot{\gamma}}}(t')\, dt'$$

Maxwell Model (integral version)

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left(\frac{\eta_0}{\lambda}\right) e^{-(t-t')/\lambda}\, \underline{\underline{\dot{\gamma}}}(t')\, dt'$$

Two parameters:

Zero-shear viscosity η_0 – gives the value of the steady shear viscosity

Relaxation time λ - quantifies how fast memory fades