We've seen that including terms that invoke past deformations (fluid memory) can improve the constitutive predictions we make.

This same class of models can be derived in differential form, beginning with the idea of combining viscous and elastic effects.

## The Maxwell Models

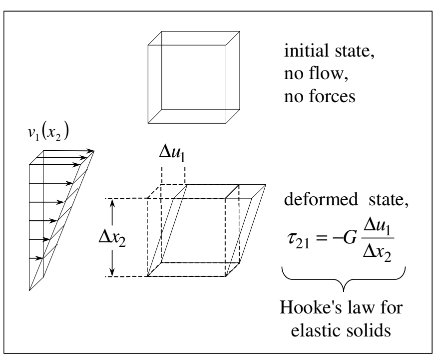
The basic Maxwell model is based on the observation that at long times viscoelastic materials behave like Newtonian liquids, while at short times they behave like elastic solids.

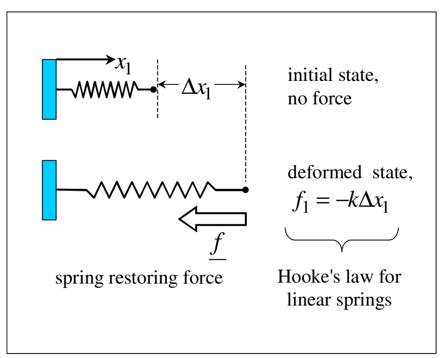
$$\tau_{21} + \frac{\mu}{G}\frac{\partial \tau_{21}}{\partial t} = -\mu\dot{\gamma}_{21}$$



*When we discussed SAOS material functions, we introduced Hooke's Law*

**Hooke's Law for elastic solids** $\tau_{21} = -G\gamma_{21}$




*Similar to the linear spring law*

Hooke's Law in shear: $\tau_{21} = -G\gamma_{21}$

Hooke's Law in general:

$$\underline{\underline{\tau}} = -G\underline{\underline{\gamma}}(t_{ref}, t)$$

$$\underline{\underline{\gamma}}(t_{ref}, t) \equiv \nabla \underline{u} + (\nabla \underline{u})^T$$

*Infinitesimal strain tensor*

---

The components of $\underline{\underline{\gamma}}(t_{ref}, t)$ are related to the components of $\underline{\underline{\dot{\gamma}}}(t')$ term by term:

$$\gamma_{21}(t_{ref}, t) = \int_{t_{ref}}^{t} \dot{\gamma}_{21}(t')dt'$$

$$\gamma_{pk}(t_{ref}, t) = \int_{t_{ref}}^{t} \dot{\gamma}_{pk}(t')dt'$$



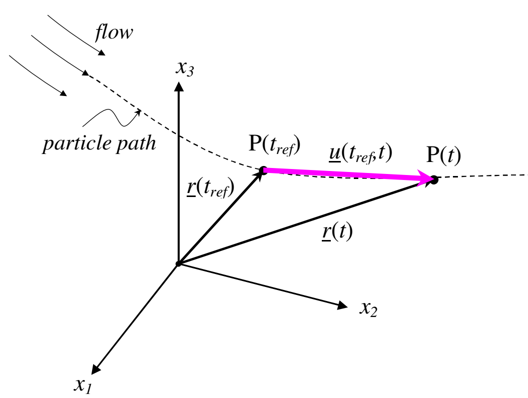
## Deformation (strain)

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

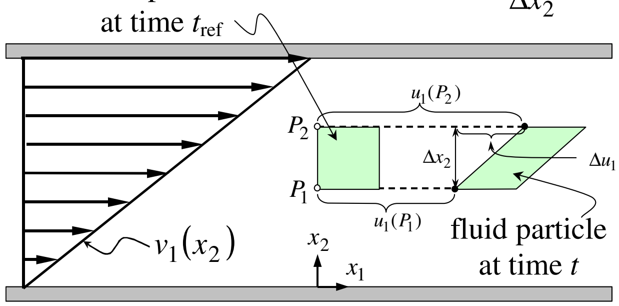
$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123}$$

$$\gamma_{21}(t_{ref}, t) \equiv \frac{\partial u_1}{\partial x_2}$$ **Shear strain**

$$\underline{u}(t_{ref}, t) \equiv \underline{r}(t) - \underline{r}(t_{ref})$$ **Displacement function**



## Physical interpretation of strain in shear




$$\gamma_{21}(t_{ref},t) = \frac{\Delta u_1}{\Delta x_2}$$

The strain is the inverse of the slope of the side of the deformed particle.

The strain is related to the *change of shape* of the deformed particle.



## Deformation in shear flow (strain)

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\gamma_{21}(t_{ref},t) \equiv \frac{\partial u_1}{\partial x_2}$$ **Shear strain**

$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123} = \begin{pmatrix} x_1(t_{ref}) + (t - t_{ref})\dot{\gamma}_0 x_2 \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t - t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

**Displacement function**

## Deformation in shear flow (strain)

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t-t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\nabla \underline{u} = \begin{pmatrix} 0 & 0 & 0 \\ (t-t_{ref})\dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

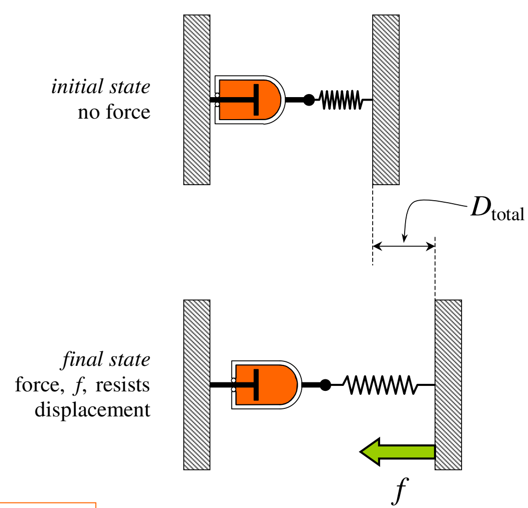
$$\underline{\underline{\gamma}} = \nabla \underline{u} + \left(\nabla \underline{u}\right)^T = \begin{pmatrix} 0 & (t-t_{ref})\dot{\gamma}_0 & 0 \\ (t-t_{ref})\dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Infinitesimal strain tensor in shear



## Maxwell's model combines viscous and elastic responses

Spring (elastic) and dashpot (viscous) in series:




Displacements are additive:

$$D_{total} = D_{spring} + D_{dashpot}$$

**In the spring:** $f = -G_{sp} D_{spring}$

**In the dashpot:** $f = -\mu \dfrac{dD_{dash}}{dt}$

$$D_{total} = D_{spring} + D_{dash}$$

$$\frac{dD_{total}}{dt} = \frac{dD_{spring}}{dt} + \frac{dD_{dash}}{dt}$$

$$= -\frac{1}{G_{sp}}\frac{df}{dt} - \frac{1}{\mu} f$$

$$\boxed{f + \frac{\mu}{G_{sp}}\frac{df}{dt} = -\mu \frac{dD_{total}}{dt}}$$



$$f + \frac{\mu}{G_{sp}}\frac{df}{dt} = -\mu \frac{dD_{total}}{dt}$$

**By analogy:**

$$\tau_{21} + \frac{\eta_0}{G}\frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21} \quad \text{shear}$$

$$\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \underline{\underline{\dot{\gamma}}} \quad \text{all flows}$$

| | | |
|---|---|---|
| Two parameter model: | $\lambda = \dfrac{\eta_0}{G}$ | *Relaxation time* |
| | $\eta_0$ | *Viscosity* |

## The Maxwell Model

In shear: $\tau_{21} + \frac{\eta_0}{G}\frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$

In general flows: $\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$

| | | |
|---|---|---|
| Two parameter model: | $\lambda = \frac{\eta_0}{G}$ | *Relaxation time* |
| | $\eta_0$ | *Viscosity* |



## The Maxwell Model

In shear: $\tau_{21} + \frac{\eta_0}{G}\frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$

In general flows: $\underline{\underline{\tau}} + \frac{\eta_0}{G}\frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$

**Solving this differential equation for the stress yields the integral equation that we introduced previously.**

| | | |
|---|---|---|
| Two parameter model: | $\lambda = \frac{\eta_0}{G}$ | *Relaxation time* |
| | $\eta_0$ | *Viscosity* |

## What are the predictions of the Maxwell model?

Need to check the predictions to see if what we have done is worth keeping.

**Predictions:**

- Steady shear
- Steady elongation
- Start-up of steady shear
- Step shear strain
- Small-amplitude oscillatory shear