3. Tensor – *the indeterminate vector product of two (or more) vectors*

e.g.: stress $\underline{\underline{\tau}}$
velocity gradient $\dot{\underline{\underline{\gamma}}}$

– *tensors may be constant or may be variable*

Definitions

dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

$\underline{a}\,\underline{d}$ dyad

$\underline{\underline{A}}$ general representation of a tensor

Laws of Algebra for Indeterminate Product of Vectors:

NO commutative $\underline{a}\,\underline{v} \neq \underline{v}\,\underline{a}$

yes associative $\underline{b}\,(\underline{a}\,\underline{v}) = (\underline{b}\,\underline{a})\,\underline{v} = \underline{b}\,\underline{a}\,\underline{v}$

yes distributive $\underline{a}\,(\underline{v}+\underline{w}) = \underline{a}\,\underline{v} + \underline{a}\,\underline{w}$

How can we represent tensors with respect to a chosen coordinate system? Just follow the rules of tensor algebra

$$\underline{a}\,\underline{m} = (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)(m_1\hat{e}_1 + m_2\hat{e}_2 + m_3\hat{e}_3)$$

$$= a_1\hat{e}_1 m_1\hat{e}_1 + a_1\hat{e}_1 m_2\hat{e}_2 + a_1\hat{e}_1 m_3\hat{e}_3 +$$

$$a_2\hat{e}_2 m_1\hat{e}_1 + a_2\hat{e}_2 m_2\hat{e}_2 + a_2\hat{e}_2 m_3\hat{e}_3 +$$

$$a_3\hat{e}_3 m_1\hat{e}_1 + a_3\hat{e}_3 m_2\hat{e}_2 + a_3\hat{e}_3 m_3\hat{e}_3$$

$$= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k\hat{e}_k m_w\hat{e}_w$$

$$= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k m_w\ \hat{e}_k\hat{e}_w$$

Any tensor may be written as the sum of 9 dyadic products of basis vectors

What about $\underline{\underline{A}}$? Same.

$$\underline{\underline{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\ \hat{e}_i\hat{e}_j$$

Einstein notation for tensors: *drop the summation sign; every double index implies a summation sign has been dropped.*

$$\underline{\underline{A}} = A_{ij}\ \hat{e}_i\hat{e}_j = A_{pk}\ \hat{e}_p\hat{e}_k$$

Reminder: the initial choice of subscript letters is arbitrary

How can we use Einstein Notation to calculate dot products between vectors and tensors?

It's the same as between vectors.

$$\underline{a} \cdot \underline{b} =$$
$$\underline{a} \cdot \underline{u}\,\underline{v} =$$
$$\underline{b} \cdot \underline{\underline{A}} =$$

Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index

2. Drop the summation signs

3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:

•In Einstein notation, the presence of repeated indices implies a missing summation sign

•The choice of initial index (i, m, p, etc.) is arbitrary - it merely indicates which indices change together