We desire a strain tensor that accurately captures large-strain deformation without being affected by rigid-body rotation.

$\nabla \underline{u}$

$\underline{\underline{\gamma}}$

$\underline{\underline{F}}$

$\underline{\underline{F}}^{-1}$

All these strain measures include both deformation and orientation

We can separate the deformation and orientation information in $\underline{\underline{F}}$ and $\underline{\underline{F}}^{-1}$ using a technique called *polar decomposition*.

Polar Decomposition Theorem

Any tensor for which an inverse exists has two unique decompositions:

$$\underline{\underline{A}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$
$$= \underline{\underline{V}} \cdot \underline{\underline{R}}$$

Pure rotation tensor

$$\underline{\underline{U}} = \left(\underline{\underline{A}}^T \cdot \underline{\underline{A}}\right)^{\frac{1}{2}}$$

$$\underline{\underline{V}} = \left(\underline{\underline{A}} \cdot \underline{\underline{A}}^T\right)^{\frac{1}{2}}$$

$$\underline{\underline{R}} = \underline{\underline{A}} \cdot \left(\underline{\underline{A}}^T \cdot \underline{\underline{A}}\right)^{-\frac{1}{2}} = \underline{\underline{A}} \cdot \underline{\underline{U}}^{-1}$$

$\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$

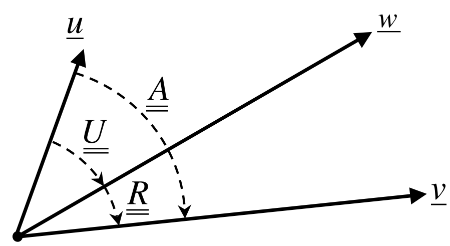
Orthogonal tensor

$\underline{\underline{U}}, \underline{\underline{V}}$

Symmetric, nonsingular tensors

EXAMPLE: Calculate the right stretch tensor and rotation tensor for a given tensor. Calculate the angle through which $\underline{\underline{R}}$ rotates the vector $\underline{u}$.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 2 \\ 2 & 0 & 0 \end{pmatrix}_{xyz} \qquad \underline{u} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_{xyz}$$



We have partially isolated the effect of rotation through polar decomposition.

$$\underline{\underline{A}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

rotation tensor ($\underline{\underline{R}}$)

left stretch tensor ($\underline{\underline{V}}$)

right stretch tensor ($\underline{\underline{U}}$)

original (strain) tensor ($\underline{\underline{A}}$)

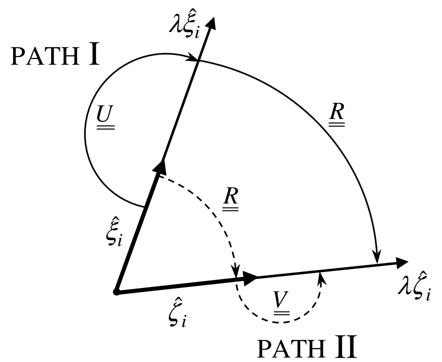
We can further isolate stretch from rotation by considering the *eigenvectors* of $\underline{\underline{U}}$ and $\underline{\underline{V}}$.

$$\underline{\underline{U}} \cdot \hat{\xi}_k = \lambda_k \hat{\xi}_k$$

$$\underline{\underline{V}} \cdot \hat{\zeta}_j = \nu_j \hat{\zeta}_j$$

eigenvectors — eigenvalues

Physical Interpretation



$$\underline{\underline{R}} \cdot \hat{\xi}_n = \hat{\zeta}_n$$

$$\lambda_n = \nu_n$$

Finite Strain Tensors

proposed deformation tensors; contain stretch and rotation

$\underline{\underline{A}}$	$\underline{\underline{V}}^2$	$\underline{\underline{U}}^2$
$\underline{\underline{F}}$	$\underline{\underline{F}} \cdot \underline{\underline{F}}^T$	$\underline{\underline{F}}^T \cdot \underline{\underline{F}}$
$\underline{\underline{F}}^T$	$\underline{\underline{F}}^T \cdot \underline{\underline{F}}$	$\underline{\underline{F}} \cdot \underline{\underline{F}}^T$
$\underline{\underline{F}}^{-1}$	$\underline{\underline{F}}^{-1} \cdot \left(\underline{\underline{F}}^{-1}\right)^T$	$\left(\underline{\underline{F}}^{-1}\right)^T \cdot \underline{\underline{F}}^{-1}$
$\left(\underline{\underline{F}}^{-1}\right)^T$	$\left(\underline{\underline{F}}^{-1}\right)^T \cdot \underline{\underline{F}}^{-1}$	$\underline{\underline{F}}^{-1} \cdot \left(\underline{\underline{F}}^{-1}\right)^T$

proposed deformation tensors; contain stretch of eigenvectors, **BUT NO ROTATION**

Cauchy tensor $\underline{\underline{C}} \equiv \underline{\underline{F}} \cdot \underline{\underline{F}}^T$

Finger tensor $\underline{\underline{C}}^{-1} \equiv \left(\underline{\underline{F}}^{-1}\right)^T \cdot \underline{\underline{F}}^{-1}$

EXAMPLE: Calculate stress predicted in rigid-body rotation by a finite-strain Hooke's law.

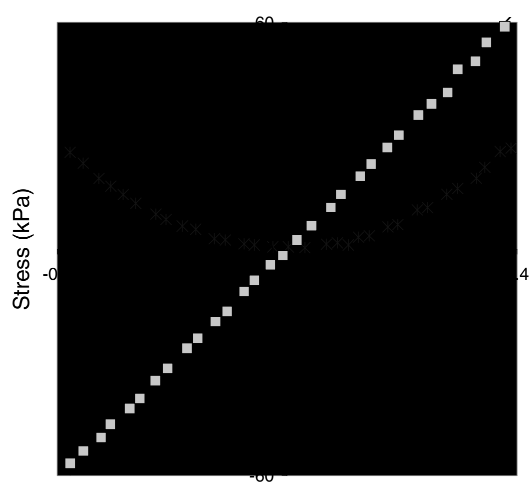
$$\underline{\underline{\tau}} = G\underline{\underline{C}}^{-1}(t,0)$$

EXAMPLE: Calculate stress predicted in shear by a finite-strain Hooke's law. Compare with experimental results.

$$\underline{\underline{\tau}} = G\underline{\underline{C}}^{-1}(t,0)$$

Compare Finite-Strain Hooke's Law with Observations

NOTE: *for the first time we have predicted nonzero normal stresses in shear.*



$$\gamma = \dot{\gamma}_0 t$$

Figure 9.6, p. 325 DeGroot; solid rubber

Now, let's fix the Maxwell model.

Integral Maxwell model (rate version):

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} \underline{\underline{\dot{\gamma}}}(t')\, dt'$$

GLVE model (strain version):

$$\begin{cases} \underline{\underline{\tau}} = +\int_{-\infty}^{t} M(t-t')\, \underline{\underline{\gamma}}(t,t')\, dt' \\ M(t-t') \equiv \dfrac{\partial G(t-t')}{\partial t'} \end{cases}$$

Integral Maxwell model (strain version):

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{\gamma}}(t,t')\, dt'$$

Lodge model

Integral Maxwell model (strain version):

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{\gamma}}(t,t')\, dt'$$

substitute (-Finger tensor) for infinitesimal strain tensor $\quad -\underline{\underline{C}}^{-1}(t',t)$

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

A finite-strain, viscoelastic constitutive equation

what does it predict?

EXAMPLE: Calculate the material functions of steady shear flow for the Lodge model.

$$\text{Lodge Model:} \quad \underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

Lodge model