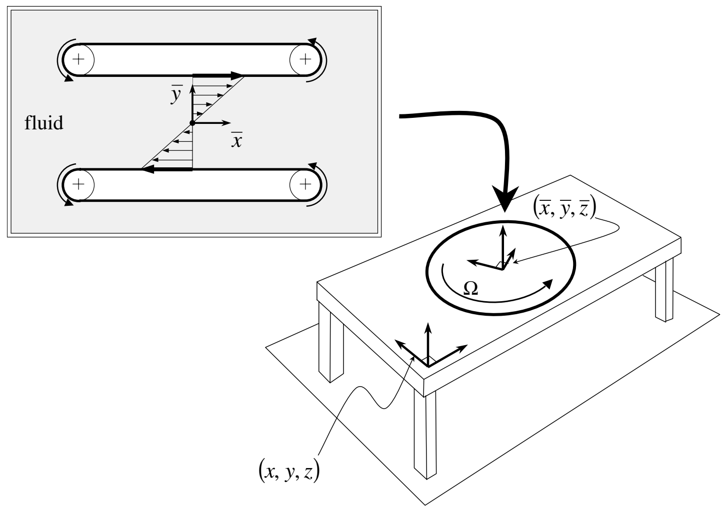
EXAMPLE: Does the Lodge model pass the test of objectivity posed by the turntable example? (remember, the GLVE failed this test)



Turntable Example

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

$$\underline{\underline{F}}^{-1}(t',t) \equiv \frac{\partial \underline{r}}{\partial \underline{r}'} = \frac{\partial r_m}{\partial r_j'} \hat{e}_j \hat{e}_m = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz}$$

$$\underline{r} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}_{xyz} = \begin{pmatrix} \bar{x}' + \dot{\gamma}_0 (t - t') \bar{y}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix}_{xyz}$$

Deformation in shear flow (strain)

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123} \qquad \gamma_{21}(t_{ref}, t) \equiv \frac{\partial u_1}{\partial x_2} \quad \textbf{Shear strain}$$

$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123} = \begin{pmatrix} x_1(t_{ref}) + (t - t_{ref})\dot{\gamma}_0 x_2 \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{u}(t_{ref}, t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t - t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

Displacement function

Turntable Example

Lodge Model: $\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t', t)\, dt'$

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\overline{xyz}}$$

Lodge prediction: rotating frame

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\overline{xyz}} dt'$$

Lodge turntable - from stationary frame

$$\underline{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}_{xyz}=\begin{pmatrix}x_0+(y'-y_0)\left[-SC'+CS'+CC'\gamma\right]+(x'-x_0)\left[SS'+CC'-CS'\gamma\right]\\ y_0+(y'-y_0)\left[C'C+S'S+SC'\gamma\right]+(x'-x_0)\left[-CS'+SC'-SS'\gamma\right]\\ z'\end{pmatrix}_{xyz}$$

$$S=\sin\Omega t$$
$$S'=\sin\Omega t'$$
$$C=\cos\Omega t$$
$$C'=\cos\Omega t'$$
$$\gamma=\dot{\gamma}_0(t-t')$$

Now, calculate $\underline{\underline{F}}^{-1}$ and $\underline{\underline{C}}^{-1}$.

$$\underline{\underline{F}}^{-1}(t',t)\equiv\frac{\partial\underline{r}}{\partial\underline{r}'}=\frac{\partial r_m}{\partial r_j'}\hat{e}_j\hat{e}_m=\begin{pmatrix}\frac{\partial x}{\partial x'}&\frac{\partial y}{\partial x'}&\frac{\partial z}{\partial x'}\\ \frac{\partial x}{\partial y'}&\frac{\partial y}{\partial y'}&\frac{\partial z}{\partial y'}\\ \frac{\partial x}{\partial z'}&\frac{\partial y}{\partial z'}&\frac{\partial z}{\partial z'}\end{pmatrix}_{xyz}$$

$$\underline{\underline{C}}^{-1}\equiv\left(\underline{\underline{F}}^{-1}\right)^T\cdot\underline{\underline{F}}^{-1}$$

Result:

$$\underline{\underline{C}}^{-1}(t',t)=\begin{pmatrix}1-2CS\gamma+C^2\gamma^2&(C^2-S^2)\gamma+SC\gamma^2&0\\(C^2-S^2)\gamma+SC\gamma^2&1+2CS\gamma+S^2\gamma^2&0\\0&0&1\end{pmatrix}_{xyz}$$

Lodge Model prediction in stationary frame:

$$\underline{\underline{\tau}}=-\int_{-\infty}^{t}\frac{\eta_0}{\lambda^2}e^{-\frac{(t-t')}{\lambda}}\begin{pmatrix}1-2CS\gamma+C^2\gamma^2&(C^2-S^2)\gamma+SC\gamma^2&0\\(C^2-S^2)\gamma+SC\gamma^2&1+2CS\gamma+S^2\gamma^2&0\\0&0&1\end{pmatrix}_{xyz}dt'$$

$$S=\sin\Omega t\qquad C=\cos\Omega t$$
$$S'=\sin\Omega t'\qquad C'=\cos\Omega t'$$
$$\gamma=\dot{\gamma}_0(t-t')$$

To compare to previous result, must consider shear coordinate system, e.g. *t=0*

Lodge prediction: stationary frame, t=0

IDENTICAL

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\overline{xyz}} dt'$$

Lodge prediction: rotating frame

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{\overline{xyz}} dt'$$

Differential Lodge Equation (Upper Convected Maxwell Model)

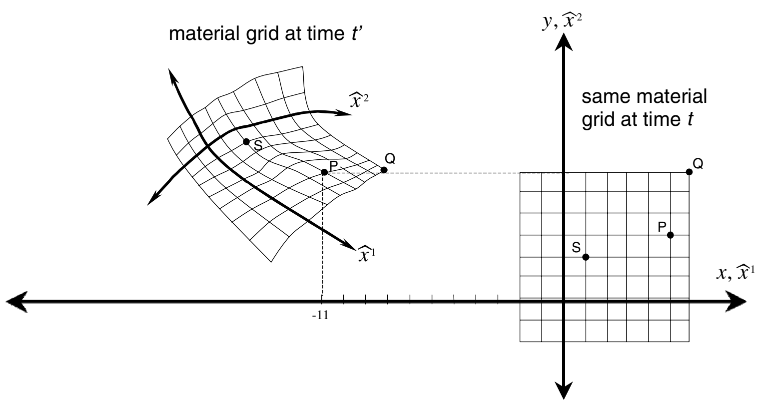
$$\underline{\underline{\tau}} + \lambda \overset{\triangledown}{\underline{\underline{\tau}}} = -\eta_0 \underline{\underline{\dot{\gamma}}}$$

$$\overset{\triangledown}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} - (\nabla \underline{v})^T \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{v}$$

upper-convected time derivative

$$\frac{D\underline{\underline{\tau}}}{Dt} \equiv \frac{\partial \underline{\underline{\tau}}}{\partial t} + \underline{v} \cdot \nabla \underline{\underline{\tau}}$$

The *Upper-Convected time derivative* can be understood to be the time derivative calculated in a coordinate system that is translating and deforming with the fluid (see text).



Other Convected Derivatives

upper-convected time derivative

$$\overset{\nabla}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} - (\nabla \underline{v})^T \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{v}$$

lower-convected time derivative

$$\overset{\Delta}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} + \nabla \underline{v} \cdot \underline{\underline{\tau}} + \underline{\underline{\tau}} \cdot (\nabla \underline{v})^T$$

Corotational time derivative

$$\overset{\circ}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} + \frac{1}{2}\left(\underline{\underline{\omega}} \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \underline{\underline{\omega}}\right)$$

$$\underline{\underline{\omega}} \equiv \nabla \underline{v} - (\nabla \underline{v})^T$$

Lodge Model:
(upper-convected Maxwell)

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

Cauchy-Maxwell Model:
(lower-convected Maxwell)

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}(t,t')\, dt'$$

Lodge Rubberlike Liquid Model:

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} M(t-t')\, \underline{\underline{C}}^{-1}(t',t)\, dt'$$

Approaches to finite-strain constitutive equations

non-objective time derivative → replace with $\overset{\Delta}{\underline{\underline{\tau}}}, \overset{\nabla}{\underline{\underline{\tau}}}$ or other time derivatives

equivalent:

differential Maxwell model

$$\underline{\underline{\tau}} + \lambda \frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_o \dot{\underline{\underline{\gamma}}}$$

integral Maxwell model

$$\underline{\underline{\tau}} = \int_{-\infty}^{t} \frac{\eta_o}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} \underline{\underline{\gamma}}(t,t')\, dt'$$

non-objective strain measure → replace with $\underline{\underline{C}}^{-1}$, $\underline{\underline{C}}$ or other strain measures