3. Tensor – *(continued)*

Definitions

Scalar product of two tensors

$$\underline{\underline{A}} : \underline{\underline{M}} = A_{ip}\hat{e}_i\hat{e}_p : M_{km}\hat{e}_k\hat{e}_m$$

$$= A_{ip}M_{km}\hat{e}_i\hat{e}_p : \hat{e}_k\hat{e}_m$$

carry out the dot products indicated

$$= A_{ip}M_{km}\left(\hat{e}_p \cdot \hat{e}_k\right)\left(\hat{e}_i \cdot \hat{e}_m\right)$$

$$= A_{ip}M_{km}\delta_{pk}\delta_{im}$$

"p" becomes "k"
"i" becomes "m"

$$= A_{mk}M_{km}$$

But, what is a tensor really?

A tensor is a handy representation of a *Linear Vector Function*

scalar function: $y = f(x) = x^2 + 2x + 3$

a mapping of values of x onto values of y

vector function: $\underline{w} = f(\underline{v})$

a mapping of vectors of $\underline{v}$ into vectors $\underline{w}$

How do we express a vector function?

What is a linear function?

Linear, in this usage, has a precise, mathematical definition.

Linear functions (scalar and vector) have the following two properties:

$$f(\lambda x) = \lambda f(x)$$
$$f(x+w) = f(x) + f(w)$$

It turns out . . .

Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).

Tensors are *Linear Vector Functions*

Let $f(\underline{a}) = \underline{b}$ be a linear vector function.

We can write $\underline{a}$ in Cartesian coordinates.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$f(\underline{a}) = f(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = \underline{b}$$

Using the linear properties of f, we can distribute the function action:

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)} + a_2 \underbrace{f(\hat{e}_2)} + a_3 \underbrace{f(\hat{e}_3)} = \underline{b}$$

These results are just vectors, we will name them $\underline{v}$, $\underline{w}$, and $\underline{m}$.

Tensors are *Linear Vector Functions* (continued)

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)}_{\underline{v}} + a_2 \underbrace{f(\hat{e}_2)}_{\underline{w}} + a_3 \underbrace{f(\hat{e}_3)}_{\underline{m}} = \underline{b}$$

$$f(\underline{a}) = a_1 \underline{v} + a_2 \underline{w} + a_3 \underline{m} = \underline{b}$$

Now we note that the coefficients a_i may be written as,

$$a_1 = \underline{a} \cdot \hat{e}_1 \quad a_2 = \underline{a} \cdot \hat{e}_2 \quad a_3 = \underline{a} \cdot \hat{e}_3$$

Substituting,

$$f(\underline{a}) = \underline{a} \cdot \hat{e}_1 \, \underline{v} + \underline{a} \cdot \hat{e}_2 \, \underline{w} + \underline{a} \cdot \hat{e}_3 \, \underline{m} = \underline{b}$$

The indeterminate vector product has appeared!

Using the distributive law, we can factor out the dot product with $\underline{a}$:

$$f(\underline{a}) = \underline{a} \cdot \underbrace{\left(\hat{e}_1 \, \underline{v} + \hat{e}_2 \, \underline{w} + \hat{e}_3 \, \underline{m}\right)} = \underline{b}$$

This is just a tensor (the sum of dyadic products of vectors)

$$\left(\hat{e}_1 \, \underline{v} + \hat{e}_2 \, \underline{w} + \hat{e}_3 \, \underline{m}\right) \equiv \underline{\underline{M}}$$

$$f(\underline{a}) = \underline{a} \cdot \underline{\underline{M}} = \underline{b}$$

CONCLUSION: Tensor operations are convenient to use to express linear vector functions.

3. Tensor – *(continued)*

More Definitions

Identity Tensor

$$\underline{\underline{I}} = \hat{e}_i \hat{e}_i = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{I}} &= A_{ip} \hat{e}_i \hat{e}_p \cdot \hat{e}_k \hat{e}_k \\ &= A_{ip} \hat{e}_i \delta_{pk} \hat{e}_k \\ &= A_{ik} \hat{e}_i \hat{e}_k \\ &= \underline{\underline{A}} \end{aligned}$$

3. Tensor – *(continued)* More Definitions

Zero Tensor

$$\underline{\underline{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Magnitude of a Tensor

$$\left| \underline{\underline{A}} \right| \equiv + \sqrt{\frac{\underline{\underline{A}} : \underline{\underline{A}}}{2}}$$

$$\begin{aligned} \underline{\underline{A}} : \underline{\underline{A}} &= A_{ip} \hat{e}_i \hat{e}_p : A_{km} \hat{e}_k \hat{e}_m \\ &= A_{ip} A_{km} \left(\hat{e}_p \cdot \hat{e}_k \right) \left(\hat{e}_i \cdot \hat{e}_m \right) \\ &= A_{mk} A_{km} \end{aligned}$$

products across the diagonal

3. Tensor – (continued) More Definitions

Tensor Transpose

$$\underline{\underline{M}}^T = \left(M_{ik}\hat{e}_i\hat{e}_k\right)^T = M_{ik}\hat{e}_k\hat{e}_i$$

Exchange the coefficients across the diagonal

CAUTION:

$$\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ik}\hat{e}_i\hat{e}_k\cdot C_{pj}\hat{e}_p\hat{e}_j\right)^T = \left(A_{ik}C_{pj}\ \hat{e}_i\hat{e}_j\delta_{kp}\right)^T$$
$$= \left(A_{ip}C_{pj}\ \hat{e}_i\hat{e}_j\right)^T$$
$$= A_{ip}C_{pj}\ \hat{e}_j\hat{e}_i$$

It is **not** equal to: $\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ip}C_{pj}\ \hat{e}_i\hat{e}_j\right)^T \neq \cancel{A_{pi}C_{jp}\ \hat{e}_i\hat{e}_j}$

I recommend you always interchange the indices on the basis vectors rather than on the coefficients.

3. Tensor – (continued) More Definitions

Symmetric Tensor

$$\underline{\underline{M}} = \underline{\underline{M}}^T$$
$$M_{ik} = M_{ki}$$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123}$$

Antisymmetric Tensor

$$\underline{\underline{M}} = -\underline{\underline{M}}^T$$
$$M_{ik} = -M_{ki}$$

e.g.

$$\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}_{123}$$

3. Tensor – (continued) More Definitions

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

order = degree of complexity

scalars	0^{th} -order tensors	3^0
vectors	1^{st} -order tensors	3^1
tensors	2^{nd} -order tensors	3^2
higher-order tensors	3^{rd} -order tensors	3^3

Number of coefficients needed to express the tensor in 3D space

3. Tensor – (continued) More Definitions

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

vectors: $|\underline{v}| = v$ The magnitude of a vector is a scalar associated with the vector

It is independent of coordinate system, i.e. it is an invariant.

tensors: $\underline{\underline{A}}$ There are three invariants associated with a second-order tensor.

Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\,\underline{\underline{A}} = tr\,\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace\,\underline{\underline{A}} = A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = A_{pk} A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = A_{pj} A_{jh} A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.