5. Curvilinear Coordinates

Cylindrical	$\bar{r},\theta,z$	$\hat{e}_{\bar{r}},\hat{e}_\theta,\hat{e}_z$
Spherical	r,θ,ϕ	$\hat{e}_r,\hat{e}_\theta,\hat{e}_\phi$

See figures 2.11 and 2.12

These coordinate systems are ortho-normal, *but they are not constant* (they vary with position).

This causes some non-intuitive effects when derivatives are taken.

5. Curvilinear Coordinates (continued)

$$\underline{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z$$

$$\nabla\cdot\underline{v} = \nabla\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$= \left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

First, we need to write this in cylindrical coordinates.

solve for Cartesian basis vectors and substitute above

$$\begin{cases} \hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y \\ \hat{e}_\theta = -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y \\ \hat{e}_z = \hat{e}_z \end{cases}$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

substitute above

5. Curvilinear Coordinates (continued)

Result:
$$\nabla = \left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z \right)$$
$$= \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

Now, proceed:

$$\nabla \cdot \underline{v} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right)$$
$$= \hat{e}_r \frac{\partial}{\partial r} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right) +$$
$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right) +$$
$$\hat{e}_z \frac{\partial}{\partial z} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right)$$

(We cannot use Einstein notation because these are not Cartesian coordinates)

5. Curvilinear Coordinates (continued)

$$\nabla \cdot \underline{v} = \hat{e}_r \frac{\partial}{\partial r} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right) +$$
$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right) +$$
$$\hat{e}_z \frac{\partial}{\partial z} \cdot \left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z \right)$$

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot v_r\hat{e}_r = \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial v_r\hat{e}_r}{\partial \theta}$$
$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y \right)$$
$$= -\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y$$
$$= \hat{e}_\theta$$

5. Curvilinear Coordinates (continued)

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}\cdot v_r\hat{e}_r = \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial v_r\hat{e}_r}{\partial\theta}$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \frac{\partial \hat{e}_r}{\partial\theta} + \hat{e}_r \frac{\partial v_r}{\partial\theta}\right)$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r\hat{e}_\theta + \hat{e}_r \frac{\partial v_r}{\partial\theta}\right)$$

$$= \frac{1}{r}v_r$$

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position..

5. Curvilinear Coordinates (continued)

Final result for divergence of a vector in cylindrical coordinates:

$$\nabla\cdot\underline{v} = \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$

$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$\nabla\cdot\underline{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$$

5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

- The basis vectors are ortho-normal
- The basis vectors are non-constant (vary with position)
- These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
- *We cannot use Einstein notation* – must use Tables in Appendix C2 (pp464-468).

6. Vector and Tensor Theorems and definitions

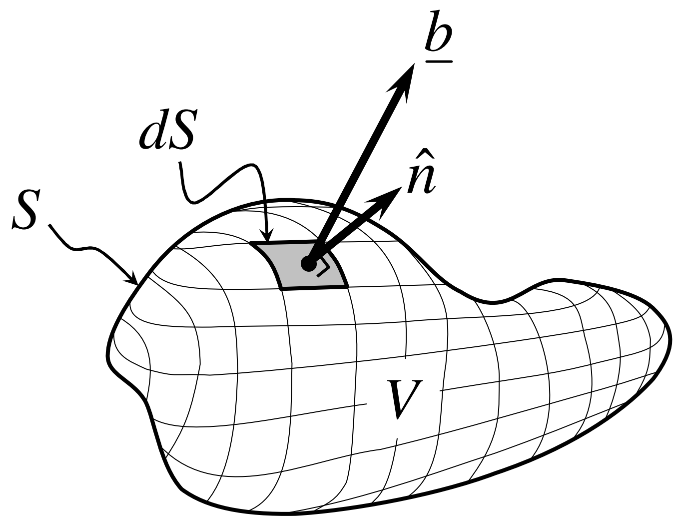
In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

Gauss Divergence Theorem

$$\iiint_V \nabla \cdot \underline{b}\, dV = \iint_S \hat{n} \cdot \underline{b}\, dS$$

outwardly directed unit normal

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.



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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

constant limits

$$I = \int_{\alpha}^{\beta} f(x,t)\, dx$$

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\alpha}^{\beta} f(x,t)\, dx$$

$$= \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t}\, dx$$

one dimension, constant limits

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6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx$$

variable limits

$$\frac{dJ}{dt} = \frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx$$

$$= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\, dx + \frac{d\beta}{dt} f(\beta,t) - \frac{d\alpha}{dt} f(\alpha,t)$$

one dimension, *variable* limits

6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \iiint_{V(t)} f(x,y,z,t)\, dV$$

$$\frac{dJ}{dt} = \frac{d}{dt}\iiint_{V(t)} f(x,y,z,t)\, dV$$

$$= \iiint_{V(t)} \frac{\partial f(x,y,z,t)}{\partial t}\, dV + \iint_{S(t)} f\left(\underline{v}_{surface} \cdot \hat{n}\right) dS$$

velocity of the surface element dS

three dimensions, *variable* limits

6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

Consider a function $f(x,y,z,t)$

$$df \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} dx + \left(\frac{\partial f}{\partial y}\right)_{xzt} dy + \left(\frac{\partial f}{\partial z}\right)_{xyt} dz + \left(\frac{\partial f}{\partial t}\right)_{xyz} dt$$

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

time rate of change of f along a chosen path

x-component of velocity along that path

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

6. Vector and Tensor Theorems *(continued)* **Substantial Derivative**

When the chosen path is the path of a fluid particle, then the space derivatives are the components of the particle velocities.

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

$$\left(\frac{df}{dt}\right)_{\substack{along \\ a\ particle \\ path}} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} v_x + \left(\frac{\partial f}{\partial y}\right)_{xzt} v_y + \left(\frac{\partial f}{\partial z}\right)_{xyt} v_z + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

$$\underline{v} \cdot \nabla f$$

$$\left(\frac{df}{dt}\right)_{\substack{along \\ a\ particle \\ path}} \equiv \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f$$

Substantial Derivative