Momentum Balance

Momentum is conserved.

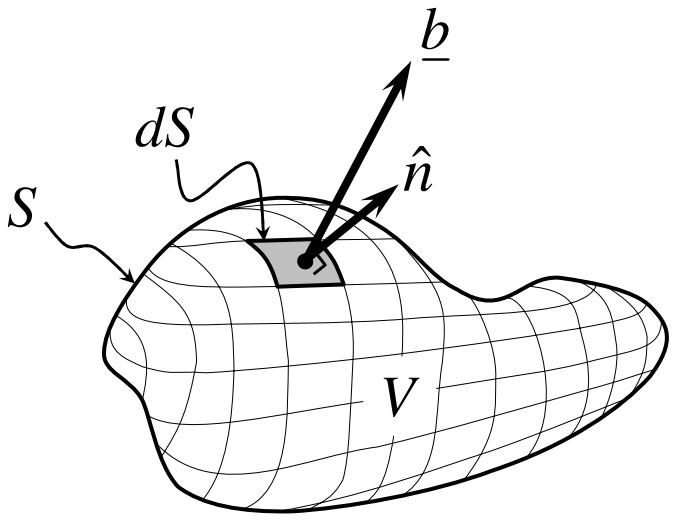
Consider an arbitrary volume V enclosed by a surface S

$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum in } V\end{pmatrix} = \begin{pmatrix}\text{net flux of}\\ \text{momentum into } V\end{pmatrix} + \begin{pmatrix}\text{sum of}\\ \text{forces on } V\end{pmatrix}$$

resembles the rate term in the mass balance

resembles the flux term in the mass balance

Forces: body (gravity) molecular forces



$$\begin{pmatrix}\text{rate of increase}\\ \text{of momentum in } V\end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \underline{v}\, dV\right) = \iiint_V \frac{\partial}{\partial t}(\rho \underline{v}) dV$$

Leibnitz rule

$$\begin{pmatrix}\text{net flux of}\\ \text{momentum into } V\end{pmatrix} = -\iint_S \hat{n}\cdot(\rho \underline{v}\,\underline{v}) dS = -\iiint_V \nabla\cdot(\rho \underline{v}\,\underline{v})\, dV$$

Gauss Divergence Theorem

Forces on *V*

Body Forces

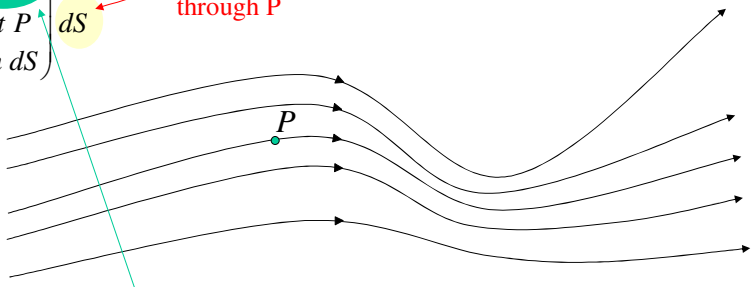
$$\begin{pmatrix}\text{force on } V\\ \text{due to } \underline{g}\end{pmatrix} = \iiint_V \rho \underline{g}\, dV$$

Molecular Forces – this is the tough one

$$\underline{f} = \begin{pmatrix} \text{stress} \\ \text{at } P \\ \text{on } dS \end{pmatrix} dS$$

choose a surface through P

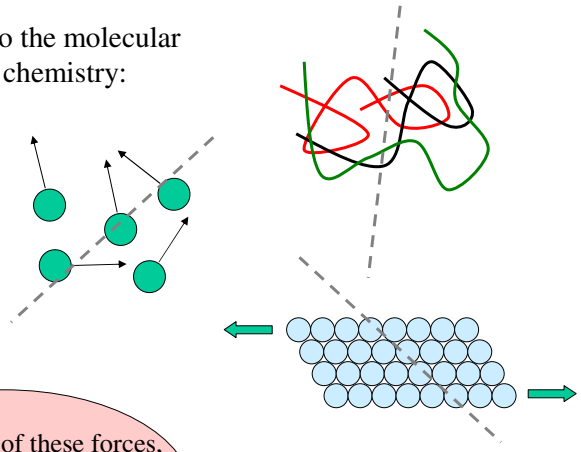
the force on that surface



We need an expression for the state of **stress** at an arbitrary point P in a flow.

Molecular Forces *(continued)*

Think back to the molecular picture from chemistry:

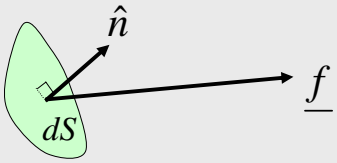


The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.

Molecular Forces *(continued)*

- We will concentrate on **expressing the molecular forces** mathematically;
- We leave to later the task of relating the resulting mathematical expression to experimental observations.

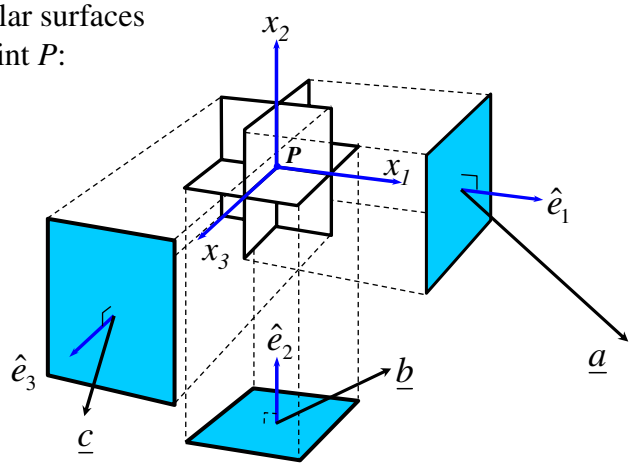
First, choose a surface:
- arbitrary shape
- small



$$\begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} = \frac{\underline{f}}{dS}$$

What is $\underline{f}$?

Consider the forces on three mutually perpendicular surfaces through point P:



Molecular Forces *(continued)*

$\underline{a}$ is stress on a "1" surface at P

a surface with unit normal $\hat{e}_1$

$\underline{b}$ is stress on a "2" surface at P

$\underline{c}$ is stress on a "3" surface at P

We can write these vectors in a Cartesian coordinate system:

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$

stress on a "1" surface in the 1-direction

Molecular Forces *(continued)*

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$
$$\underline{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$
$$= \Pi_{21}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{23}\hat{e}_3$$
$$\underline{c} = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3$$
$$= \Pi_{31}\hat{e}_1 + \Pi_{32}\hat{e}_2 + \Pi_{33}\hat{e}_3$$

$\underline{a}$ is stress on a "1" surface at P

$\underline{b}$ is stress on a "2" surface at P

$\underline{c}$ is stress on a "3" surface at P

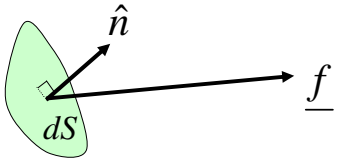
So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

Π_{pk}

Stress on a "*p*" surface in the *k*-direction

Molecular Forces *(continued)*

How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the Π_{pk}?



$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1 is force on dS in 1-direction

f_2 is force on dS in 2-direction

f_3 is force on dS in 3-direction

There are <u>three</u> Π_{pk} that relate to forces in the 1-direction: $\Pi_{11}, \Pi_{21}, \Pi_{31}$