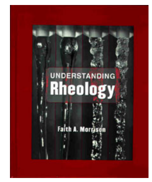
Chapter 3: Newtonian Fluids

CM4650
Polymer Rheology
Michigan Tech



Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

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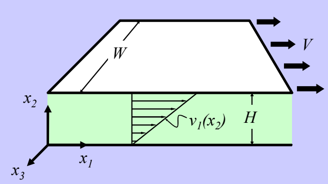
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- •Derive governing equations (mass and momentum balances
- •Solve governing equations for velocity and stress fields

QUICK START



<u>First</u>, before we get deep into derivation, let's do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.

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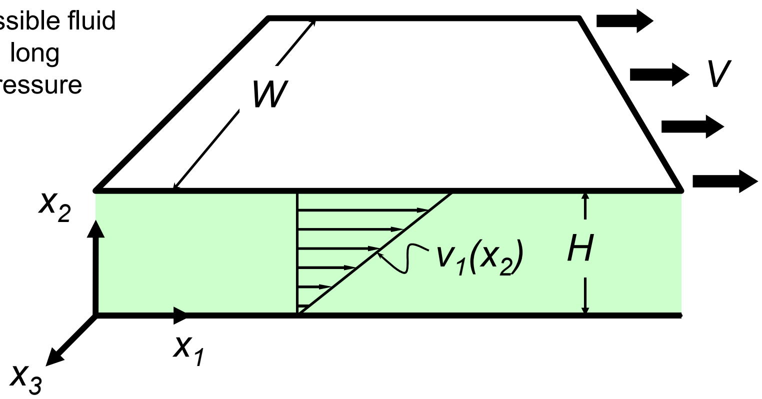
EXAMPLE: Drag flow between infinite parallel plates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$

- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure



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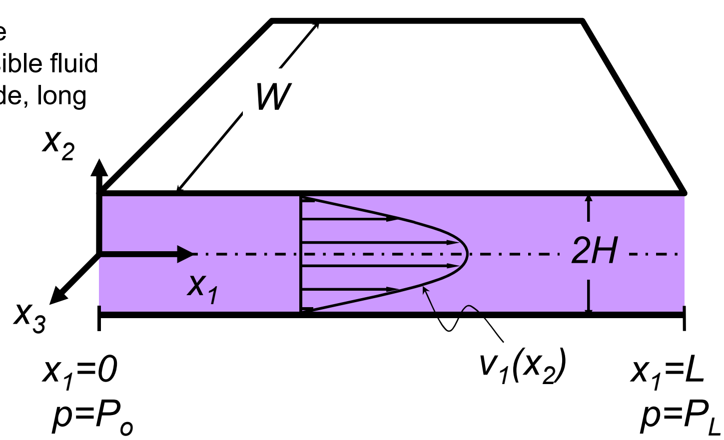
EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- steady state
- Incompressible fluid
- infinitely wide, long



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Engineering Quantities of Interest

(any flow)

In more complex flows, we can use general expressions that work in all cases.

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v})\Big|_{surface} dS$$

average velocity

$$\langle v_z \rangle = \frac{\iint_S (\hat{n} \cdot \underline{v})|_{surface} dS}{\iint_S dS}$$

Using the general formulas will help prevent errors.

Here, $\hat{n}$ is the outwardly pointing unit normal of dS; it points in the direction "through" S

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The stress tensor was invented to make the calculation of fluid stress easier.

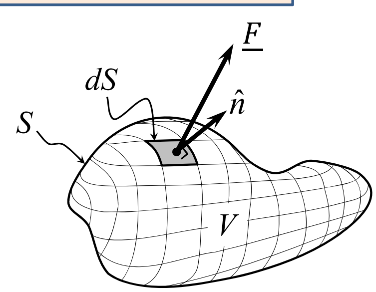
Total stress tensor, $\underline{\underline{\Pi}}$:

$$\underline{\underline{\Pi}} \equiv p\underline{\underline{I}} + \underline{\underline{\tau}}$$

(any flow, small surface)

$$\text{Force on the surface } dS = \hat{n} \cdot (-\underline{\underline{\Pi}})\, dS$$

(using the stress convention of *Understanding Rheology*)



Here, $\hat{n}$ is the outwardly pointing unit normal of dS; it points in the direction "through" S

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To get the total force on the macroscopic surface S, we integrate over the entire surface of interest.

Fluid force on the surface S

$$\underline{F} = \iint_S \Big[\hat{n} \cdot \overbrace{\left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)}^{-\underline{\underline{\Pi}}}\Big]_{surface} dS$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{xyz} = \iint_S \begin{pmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \end{pmatrix} \cdot \begin{pmatrix} -p-\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p-\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p-\tau_{zz} \end{pmatrix}_{xyz} dS$$

$\hat{n}$, $\underline{\underline{\tau}}$ and p evaluated at the surface dS

(using the stress convention of *Understanding Rheology*)

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Engineering Quantities of Interest

(any flow)

Using the general formulas will help prevent errors (like forgetting the pressure).

force **on** the surface, S

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)\right]_{surface} dS$$

z-component of force on the surface, S

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} - \underline{\underline{\tau}}\right)\right]_{surface} dS$$

(using the stress convention of *Understanding Rheology*)

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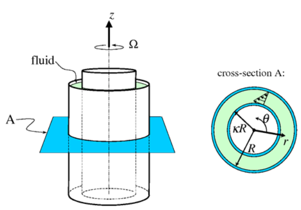
Engineering Quantities of Interest

(any flow)

Total Fluid Torque on a surface, S

$$\underline{T} = \iint_S \left[\underline{R} \times \left(\hat{n} \cdot \left(-\underline{\underline{\Pi}} \right) \right) \right]_{surface} dS$$



$\underline{R}$ is the vector from the axis of rotation to dS

(using the stress convention of *Understanding Rheology*)

Common surface shapes:

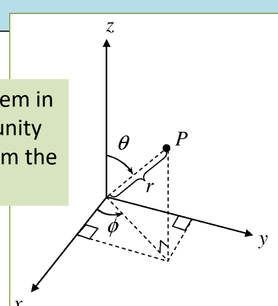
rectangular : $dS = dxdy$

circular top : $dS = r\,drd\theta$

surface of cylinder : $dS = Rd\theta dz$

sphere : $dS = (Rd\theta)\left(r \sin\theta\, d\phi\right) = R^2 \sin\theta\, d\theta d\phi$

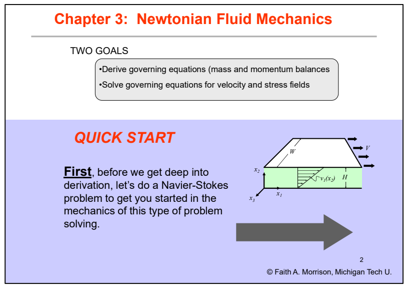
Note: spherical coordinate system in use by fluid mechanics community uses $0 < \theta < \pi$ as the angle from the z=axis to the point.



For more areas, see Exam 1 formula handout:

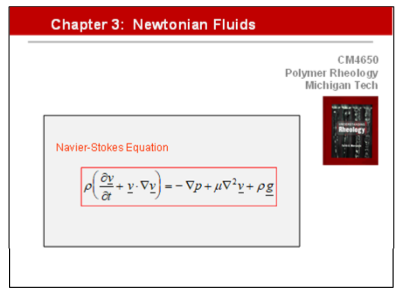
pages.mtu.edu/~fmorriso/cm4650/formula_sheet_for_exam1_2018.pdf

Review:



We got a *Quick Start* with Newtonian problem solving…

Now…
Back to exploring the origin of the equations (so we can adapt to non-Newtonian)



Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- •Derive governing equations (mass and momentum balances
- •Solve governing equations for velocity and stress fields

Mass Balance

Consider an arbitrary control volume *V* enclosed by a surface *S*

$$\begin{pmatrix} rate\ of\ increase \\ of\ mass\ in\ CV \end{pmatrix} = \begin{pmatrix} net\ flux\ of \\ mass\ into\ CV \end{pmatrix}$$

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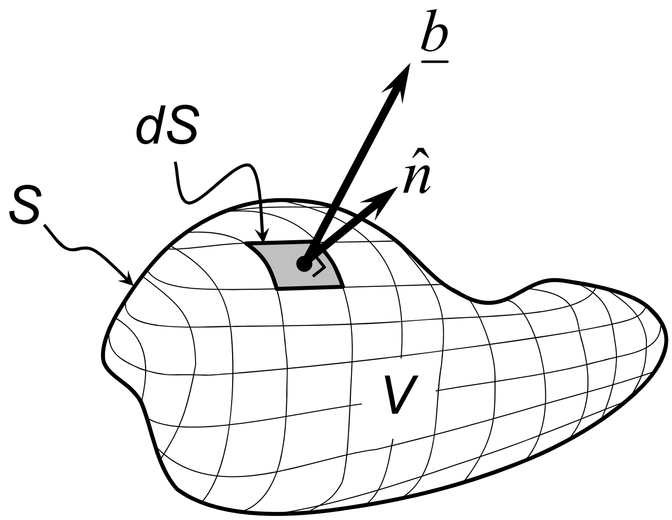
Mathematics Review Polymer Rheology



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Chapter 3: Newtonian Fluid Mechanics Polymer Rheology

Mass Balance *(continued)*

Consider an arbitrary volume *V* enclosed by a surface *S*

$$\begin{pmatrix} \text{rate of increase} \\ \text{of mass in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \, dV \right)$$

$$\begin{pmatrix} \text{net flux of} \\ \text{mass into } V \\ \text{through surface } S \end{pmatrix} = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

outwardly pointing unit normal

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Mass Balance (continued)

$$\frac{d}{dt}\left(\iiint_V \rho\, dV\right) = -\iint_S \rho\, \hat{n}\cdot \underline{v}\, dS$$

Leibnitz rule

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \hat{n}\cdot\left(\rho \underline{v}\right) dS$$

Gauss Divergence Theorem

$$= -\iiint_V \nabla\cdot\left(\rho\underline{v}\right) dV$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right)\right) dV = 0$$

Mass Balance (continued)

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right)\right) dV = 0$$

Since V is *arbitrary,*

Continuity equation:
microscopic mass balance

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\underline{v}\right) = 0$$

Mass Balance *(continued)*

Continuity equation (general fluids)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \underline{v}) = 0$$

For ρ=constant (incompressible fluids):

$$\nabla \cdot \underline{v} = 0$$

Momentum Balance

Consider an arbitrary control volume *V* enclosed by a surface *S*

Momentum is conserved.

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } CV \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{momentum into } CV \end{pmatrix} + \begin{pmatrix} \text{sum of} \\ \text{forces on } CV \end{pmatrix}$$

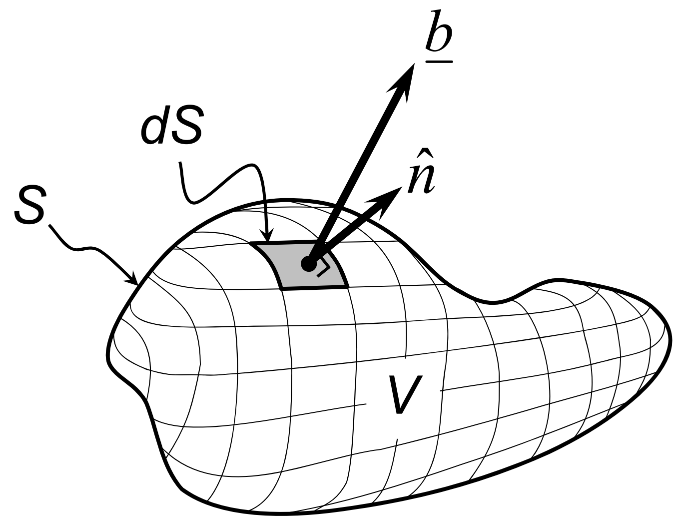
resembles the rate term in the mass balance

resembles the flux term in the mass balance

Forces: body (gravity) molecular forces

Momentum Balance

Polymer Rheology



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Momentum Balance *(continued)*

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$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \underline{v}\, dV \right)$$

$$= \iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV$$

Leibnitz rule

$$\begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} = -\iint_S \hat{n} \cdot \left(\rho \underline{v}\,\underline{v}\right) dS$$

$$= -\iiint_V \nabla \cdot \left(\rho \underline{v}\,\underline{v}\right) dV$$

Gauss Divergence Theorem

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Momentum Balance *(continued)* Polymer Rheology

Forces on *V*

Body Forces (non-contact)

$$\begin{pmatrix} force\ on\ V \\ due\ to\ \underline{g} \end{pmatrix} = \iiint_V \rho \underline{g}\ dV$$

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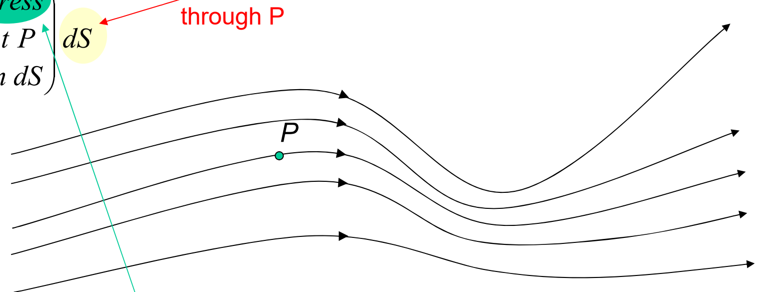
Molecular Forces (contact) – this is the tough one

$$\underline{f} = \begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS$$

choose a surface through P

the force on that surface



We need an expression for the state of **stress** at an arbitrary point P in a flow.

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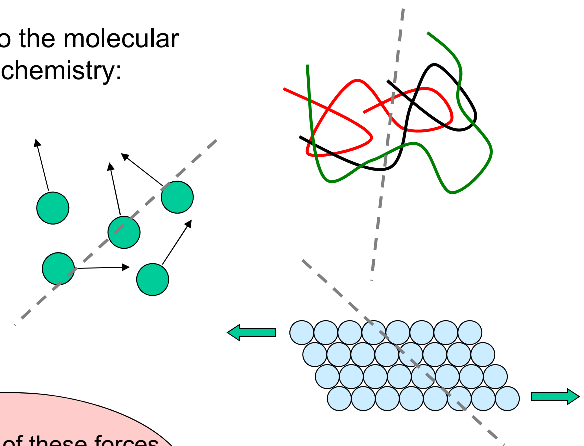
Molecular Forces *(continued)*

Think back to the molecular picture from chemistry:



The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.

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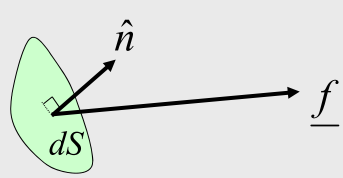
Molecular Forces *(continued)*

- We will concentrate on **expressing the molecular forces** mathematically;
- We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:

- arbitrary shape
- small



$$\begin{pmatrix} stress \\ at\ P \\ on\ dS \end{pmatrix} dS = \underline{f}$$

What is *f* ?

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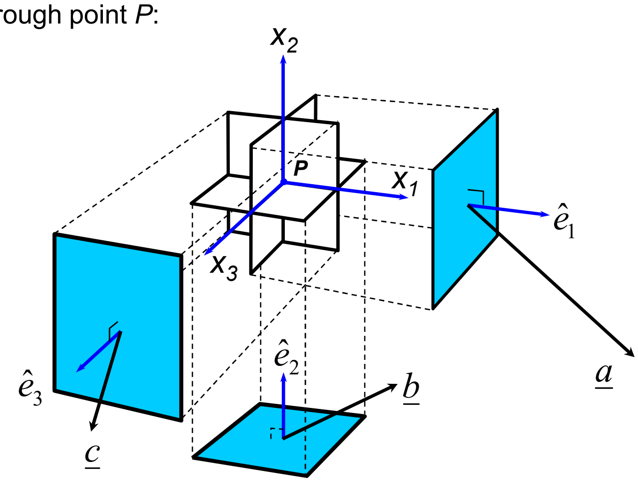
Consider the forces on three mutually perpendicular surfaces through point *P*:



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Molecular Forces *(continued)*

$\underline{a}$ is stress on a "1" surface at P

a surface with unit normal $\hat{e}_1$

$\underline{b}$ is stress on a "2" surface at P

$\underline{c}$ is stress on a "3" surface at P

We can write these vectors in a Cartesian coordinate system:

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$

stress on a "1" surface in the 1-direction

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Molecular Forces *(continued)*

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \Pi_{11}\hat{e}_1 + \Pi_{12}\hat{e}_2 + \Pi_{13}\hat{e}_3$$
$$\underline{b} = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3$$
$$= \Pi_{21}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{23}\hat{e}_3$$
$$\underline{c} = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3$$
$$= \Pi_{31}\hat{e}_1 + \Pi_{32}\hat{e}_2 + \Pi_{33}\hat{e}_3$$

$\underline{a}$	is stress on a "1" surface at P
$\underline{b}$	is stress on a "2" surface at P
$\underline{c}$	is stress on a "3" surface at P

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

Π_{pk} — Stress on a "*p*" surface in the *k*-direction

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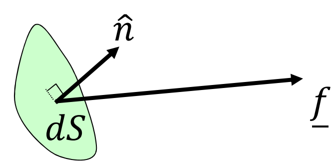
Molecular Forces *(continued)*

How can we write $\underline{f}$ (the force on an arbitrary surface *dS*) in terms of the P_{pk}?



$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1 is force on dS in 1-direction

f_2 is force on dS in 2-direction

f_3 is force on dS in 3-direction

There are three P_{pk} that relate to forces in the 1-direction: $\Pi_{11}, \Pi_{21}, \Pi_{31}$

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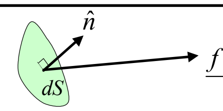
Molecular Forces *(continued)*



How can we write $\underline{f}$ (the force on an arbitrary surface dS) in terms of the quantities Π_{pk}?

$$\underline{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

f_1, the force on dS in 1-direction, can be broken into three parts associated with the three stress components:

$$\Pi_{11}, \Pi_{21}, \Pi_{31}$$

first part:

$$\underbrace{(\Pi_{11})}_{\left(\frac{force}{area}\right)} \cdot \underbrace{\begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 1-surface \end{pmatrix}}_{(area)} = \Pi_{11}\hat{n}\cdot\hat{e}_1\,dS$$

where the projection of dA onto the 1-surface is $\hat{n}\cdot\hat{e}_1\,dS$.

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Molecular Forces *(continued)*

f_1, the force on dS in 1-direction, is composed of THREE parts:

first part:
$$(\Pi_{11})\begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 1-surface \end{pmatrix} = \Pi_{11}\hat{n}\cdot\hat{e}_1\,dS$$

second part:
$$(\Pi_{21})\begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 2-surface \end{pmatrix} = \Pi_{21}\hat{n}\cdot\hat{e}_2\,dS$$

stress on a 2-surface in the 1-direction

third part:
$$(\Pi_{31})\begin{pmatrix} projection\ of \\ dA\ onto\ the \\ 3-surface \end{pmatrix} = \Pi_{31}\hat{n}\cdot\hat{e}_3\,dS$$

the sum of these three = f_1

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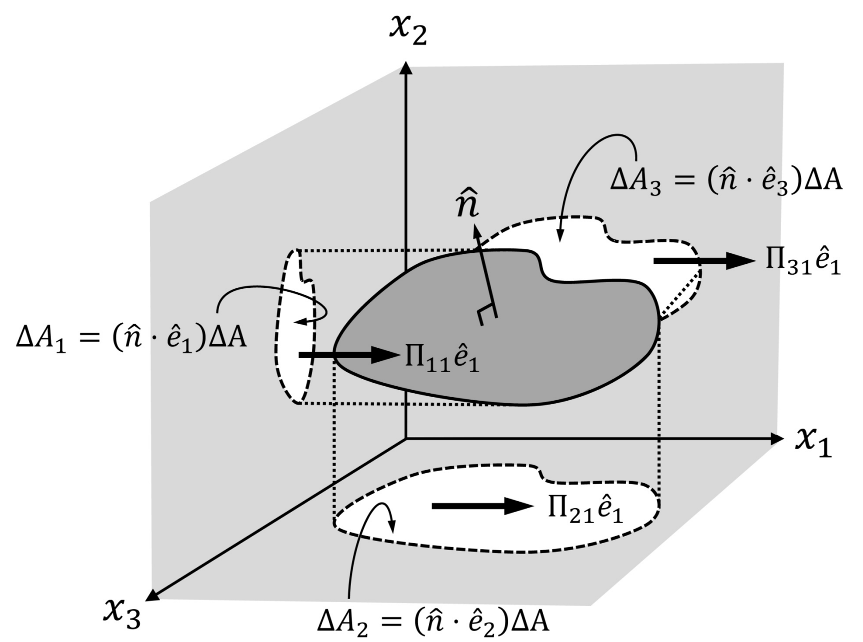

Molecular Forces *(continued)*

f_1, the force in the 1-direction on an arbitrary surface dS is composed of THREE parts.

$$f_1 = \Pi_{11}\hat{n}\cdot\hat{e}_1\, dS + \Pi_{21}\hat{n}\cdot\hat{e}_2\, dS + \Pi_{31}\hat{n}\cdot\hat{e}_3\, dS$$

stress; appropriate area

Using the distributive law:

$$f_1 = \hat{n}\cdot\left(\Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3\right) dS$$

Force in the 1-direction on an arbitrary surface dS

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Molecular Forces *(continued)*

The same logic applies in the 2-direction and the 3-direction

$$f_1 = \hat{n} \cdot \left(\Pi_{11} \hat{e}_1 + \Pi_{21} \hat{e}_2 + \Pi_{31} \hat{e}_3 \right) dS$$
$$f_2 = \hat{n} \cdot \left(\Pi_{12} \hat{e}_1 + \Pi_{22} \hat{e}_2 + \Pi_{32} \hat{e}_3 \right) dS$$
$$f_3 = \hat{n} \cdot \left(\Pi_{13} \hat{e}_1 + \Pi_{23} \hat{e}_2 + \Pi_{33} \hat{e}_3 \right) dS$$

Assembling the force vector:

$$\begin{aligned} \underline{f} &= f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3 \\ &= dS \; \hat{n} \cdot \left(\Pi_{11} \hat{e}_1 + \Pi_{21} \hat{e}_2 + \Pi_{31} \hat{e}_3 \right) \hat{e}_1 \\ &\quad + dS \; \hat{n} \cdot \left(\Pi_{12} \hat{e}_1 + \Pi_{22} \hat{e}_2 + \Pi_{32} \hat{e}_3 \right) \hat{e}_2 \\ &\quad + dS \; \hat{n} \cdot \left(\Pi_{13} \hat{e}_1 + \Pi_{23} \hat{e}_2 + \Pi_{33} \hat{e}_3 \right) \hat{e}_3 \end{aligned}$$

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Molecular Forces *(continued)*

Assembling the force vector:

$$\underline{f} = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$$

$$\begin{aligned} &= dS \; \hat{n} \cdot \left(\Pi_{11} \hat{e}_1 + \Pi_{21} \hat{e}_2 + \Pi_{31} \hat{e}_3 \right) \hat{e}_1 \\ &\quad + dS \; \hat{n} \cdot \left(\Pi_{12} \hat{e}_1 + \Pi_{22} \hat{e}_2 + \Pi_{32} \hat{e}_3 \right) \hat{e}_2 \\ &\quad + dS \; \hat{n} \cdot \left(\Pi_{13} \hat{e}_1 + \Pi_{23} \hat{e}_2 + \Pi_{33} \hat{e}_3 \right) \hat{e}_3 \end{aligned}$$

$$\begin{aligned} &= dS \; \hat{n} \cdot \left[\Pi_{11} \hat{e}_1 \hat{e}_1 + \Pi_{21} \hat{e}_2 \hat{e}_1 + \Pi_{31} \hat{e}_3 \hat{e}_1 \right. \\ &\quad + \Pi_{12} \hat{e}_1 \hat{e}_2 + \Pi_{22} \hat{e}_2 \hat{e}_2 + \Pi_{32} \hat{e}_3 \hat{e}_2 \\ &\quad \left. + \Pi_{13} \hat{e}_1 \hat{e}_3 + \Pi_{23} \hat{e}_2 \hat{e}_3 + \Pi_{33} \hat{e}_3 \hat{e}_3 \right] \end{aligned}$$

linear combination of dyadic products = **tensor**

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Molecular Forces (continued)

Assembling the force vector:

$$\underline{f} = dS\,\hat{n}\cdot\left[\Pi_{11}\hat{e}_1\hat{e}_1 + \Pi_{21}\hat{e}_2\hat{e}_1 + \Pi_{31}\hat{e}_3\hat{e}_1 + \Pi_{12}\hat{e}_1\hat{e}_2 + \Pi_{22}\hat{e}_2\hat{e}_2 + \Pi_{32}\hat{e}_3\hat{e}_2 + \Pi_{13}\hat{e}_1\hat{e}_3 + \Pi_{23}\hat{e}_2\hat{e}_3 + \Pi_{33}\hat{e}_3\hat{e}_3\right]$$

$$= dS\,\hat{n}\cdot\sum_{p=1}^{3}\sum_{m=1}^{3}\Pi_{pm}\hat{e}_p\hat{e}_m$$

$$= dS\,\hat{n}\cdot\Pi_{pm}\hat{e}_p\hat{e}_m$$

$$\underline{f} = dS\,\hat{n}\cdot\underline{\underline{\Pi}}$$

Total stress tensor (molecular stresses)

Momentum Balance (continued)

Polymer Rheology

$$\begin{pmatrix}\textit{rate of increase}\\ \textit{of momentum in } V\end{pmatrix} = \begin{pmatrix}\textit{net flux of}\\ \textit{momentum into } V\end{pmatrix} + \begin{pmatrix}\textit{sum of}\\ \textit{forces on } V\end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho\underline{v})\,dV = -\iiint_V \nabla\cdot(\rho\underline{v}\underline{v})\,dV + \iiint_V \rho\underline{g}\,dV + \begin{matrix}\textit{molecular}\\ \textit{forces}\end{matrix}$$

$$\begin{matrix}\textit{molecular}\\ \textit{forces}\end{matrix} = \iint_S \begin{pmatrix}\textit{molecular}\\ \textit{forces on}\\ dS\end{pmatrix}$$

$$= \iint_S \hat{n}\cdot\left(-\underline{\underline{\Pi}}\right)dS$$

$$= \iiint_V \nabla\cdot\left(-\underline{\underline{\Pi}}\right)dV$$

We use a stress sign convention that requires a negative sign here.

Gauss Divergence Theorem

Momentum Balance *(continued)*

Polymer Rheology

$$\begin{pmatrix} \text{rate of increase} \\ \text{of momentum in } V \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{momentum into } V \end{pmatrix} + \begin{pmatrix} \text{sum of} \\ \text{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}\left(\rho \underline{v}\right) dV = -\iiint_V \nabla \cdot \left(\rho \underline{v}\,\underline{v}\right) dV + \iiint_V \rho \underline{g}\; dV + \begin{matrix}\text{molecular}\\ \text{forces}\end{matrix}$$

$$\begin{matrix}\text{molecular}\\ \text{forces}\end{matrix} = \iint_S \begin{pmatrix} \text{molecular} \\ \text{forces on} \\ dS \end{pmatrix} = \iint_S \hat{n} \cdot \left(-\underline{\underline{\Pi}}\right) dS = \iiint_V \nabla \cdot \left(-\underline{\underline{\Pi}}\right) dV$$

UR/Bird choice: positive compression (pressure is positive)

Gauss Divergence Theorem

Momentum Balance *(continued)*

Polymer Rheology

$$\underline{F}_{\substack{on \\ surface}} = \iint_S \hat{n} \cdot \left(-\underline{\underline{\Pi}}\right) dS = \iint_S \hat{n} \cdot \left(\underline{\underline{\tilde{\Pi}}}\right) dS$$

$$\Pi_{yx} \qquad \tilde{\Pi}_{yx}$$

UR/Bird choice: fluid at lesser *y* exerts force on fluid at greater *y*

(IFM/Mechanics choice: (opposite)

Momentum Balance *(continued)*

Polymer Rheology

Final Assembly:

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of momentum in } V \end{pmatrix} = \begin{pmatrix} \textit{net flux of} \\ \textit{momentum into } V \end{pmatrix} + \begin{pmatrix} \textit{sum of} \\ \textit{forces on } V \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho \underline{v}) dV = -\iiint_V \nabla \cdot (\rho \underline{v}\underline{v})\, dV + \iiint_V \rho \underline{g}\ dV - \iiint_V \nabla \cdot \underline{\underline{\Pi}}\, dV$$

$$\iiint_V \left[\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} \right] dV = 0$$

Because *V* is *arbitrary*, we may conclude:

$$\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} = 0$$

Microscopic momentum balance

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Momentum Balance *(continued)*

Polymer Rheology

Microscopic momentum balance

$$\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot (\rho \underline{v}\underline{v}) - \rho \underline{g} + \nabla \cdot \underline{\underline{\Pi}} = 0$$

After some rearrangement:

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

$$\rho \frac{D\underline{v}}{Dt} = -\nabla \cdot \underline{\underline{\Pi}} + \rho \underline{g}$$

Equation of Motion

Now, what to do with $\underline{\underline{\Pi}}$?

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Momentum Balance *(continued)* Polymer Rheology

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

Pressure

definition: An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

$$pressure = p\,\underline{\underline{I}} = p\,\hat{e}_1\hat{e}_1 + p\,\hat{e}_2\hat{e}_2 + p\,\hat{e}_3\hat{e}_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{123}$$

Test: what is the force on a surface with unit normal $\hat{n}$?

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Momentum Balance *(continued)* Polymer Rheology

back to our question,

Now, what to do with $\underline{\underline{\Pi}}$? Pressure is part of it.

There are other, nonisotropic stresses

Extra Molecular Stresses

definition: The extra stresses are the molecular stresses that are not isotropic

$$\underline{\underline{\tau}} \equiv \underline{\underline{\Pi}} - p\,\underline{\underline{I}}$$

(other sign convention: $\underline{\underline{\tilde{\tau}}} = \underline{\underline{\tilde{\Pi}}} + p\underline{\underline{I}}$*)*

Extra stress tensor, i.e. everything complicated in molecular deformation

Now, what to do with $\underline{\underline{\tau}}$?

This becomes the central question of rheological study

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Momentum Balance *(continued)* Polymer Rheology

Stress sign convention affects any expressions with $\underline{\underline{\Pi}}, \underline{\underline{\tilde{\Pi}}}$ or $\underline{\underline{\tau}}, \underline{\underline{\tilde{\tau}}}$

$$\underline{\underline{\Pi}} \equiv \underline{\underline{\tau}} + p\,\underline{\underline{I}}$$

UR/Bird choice: fluid at lesser *y* exerts force on fluid at greater *y*

$$\underline{\underline{\tilde{\Pi}}} \equiv \underline{\underline{\tilde{\tau}}} - p\,\underline{\underline{I}}$$

(IFM/Mechanics choice: (opposite)

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Momentum Balance *(continued)* Polymer Rheology

Constitutive equations for Stress

- •are tensor equations
- •relate the velocity field to the stresses generated by molecular forces
- •are based on observations (empirical) or are based on molecular models (theoretical)
- •are typically found by trial-and-error
- •are justified by how well they work for a system of interest
- •are observed to be symmetric

$$\underline{\underline{\tau}} = f(\nabla \underline{v}, \; material\ properties)$$

Observation: the stress tensor is symmetric

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Momentum Balance *(continued)*

Polymer Rheology

Microscopic momentum balance

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla\cdot\underline{\underline{\Pi}}+\rho\underline{g}$$

Equation of Motion

In terms of the extra stress tensor:

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p-\nabla\cdot\underline{\underline{\tau}}+\rho\underline{g}$$

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):
http://www.chem.mtu.edu/~fmorriso/cm310/Navier2007.pdf

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Momentum Balance *(continued)*

Polymer Rheology

Newtonian Constitutive equation

$$\underline{\underline{\tau}}=-\mu\left(\nabla\underline{v}+\left(\nabla\underline{v}\right)^T\right)$$

- for incompressible fluids (see text for compressible fluids)
- is empirical
- may be justified for some systems with molecular modeling calculations

Note: $\underline{\underline{\tilde{\tau}}}=+\mu(\nabla\underline{v}+(\nabla\underline{v})^T)$ (IFM choice: (opposite)

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Momentum Balance *(continued)*

Polymer Rheology

How is the Newtonian Constitutive equation related to Newton's Law of Viscosity?

$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

•incompressible fluids

$$\tau_{21} = -\mu \frac{\partial v_1}{\partial x_2}$$

•incompressible fluids
•rectilinear flow (straight lines)
•no variation in x_3-direction

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Momentum Balance *(continued)*

Polymer Rheology

Back to the momentum balance . . .

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$$

Equation of Motion

$$\underline{\underline{\tau}} = -\mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids

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Momentum Balance *(continued)* Polymer Rheology

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

- incompressible fluids
- Newtonian fluids

Note: The Navier-Stokes is unaffected by the stress sign convention because neither $\underline{\underline{\tau}}$ nor $\underline{\underline{\tilde{\tau}}}$ appear.

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Momentum Balance *(continued)* Polymer Rheology

Next?

Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

Newtonian Problem Solving

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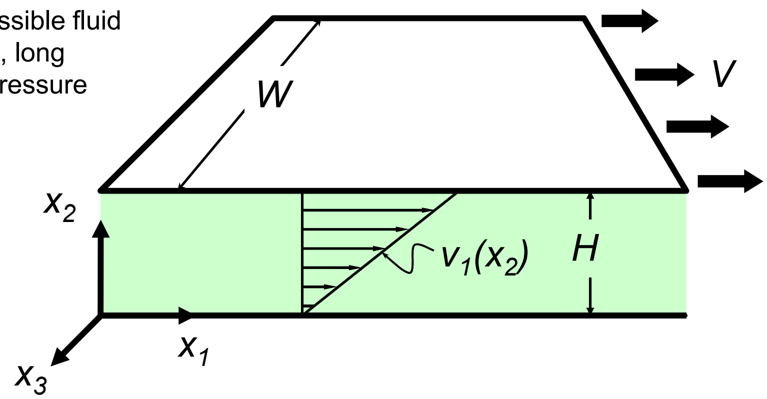
from **QUICK START**

EXAMPLE: Drag flow between infinite parallel plates

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123}$$

- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure



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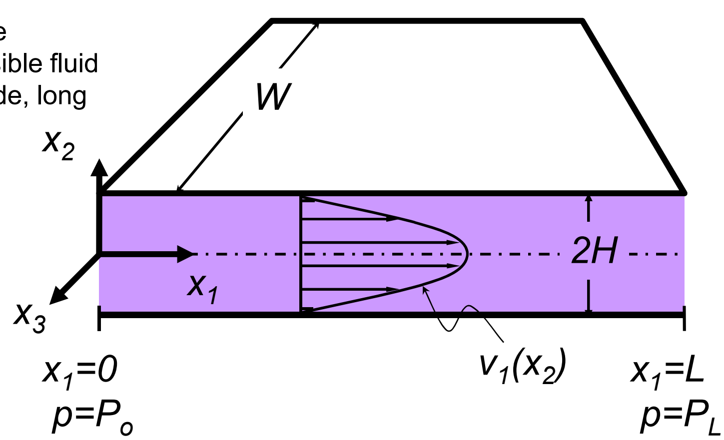
from **QUICK START**

EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- steady state
- Incompressible fluid
- infinitely wide, long



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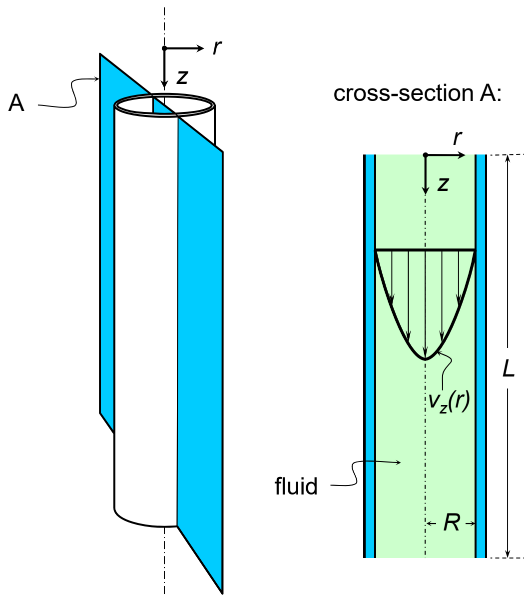
EXAMPLE: Poiseuille flow in a tube

- Newtonian
- Steady state
- incompressible fluid
- long tube



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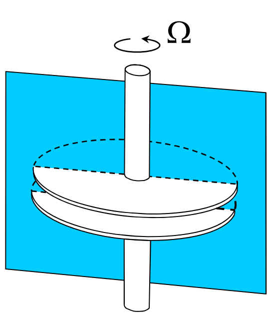
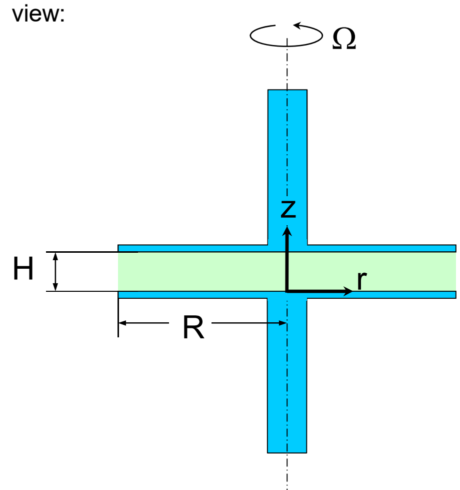
EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- incompressible fluid
- $v_\theta = zf(r)$





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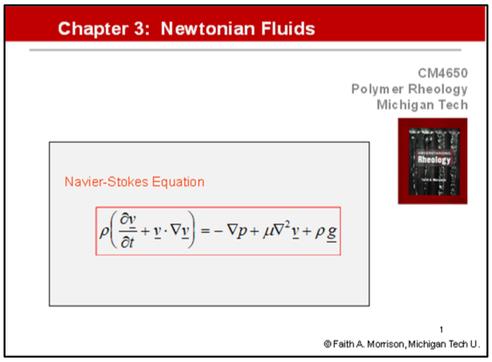
Done with
Newtonian Fluids.



Let's move on to
Standard Flows

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Chapter 4: Standard Flows

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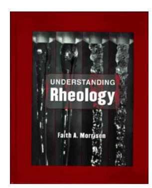
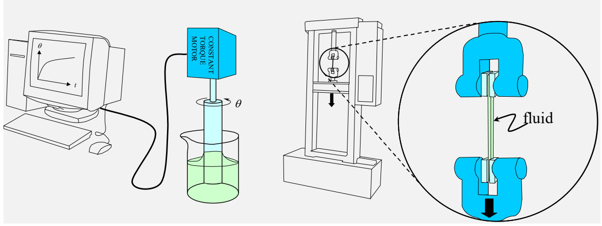


Newtonian fluids: $\underline{\underline{\tau}} = -\mu\dot{\underline{\underline{\gamma}}}$ *vs.* non-Newtonian fluids: $\underline{\underline{\tau}} \neq -\mu\dot{\underline{\underline{\gamma}}}$

How can we investigate non-Newtonian behavior?



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