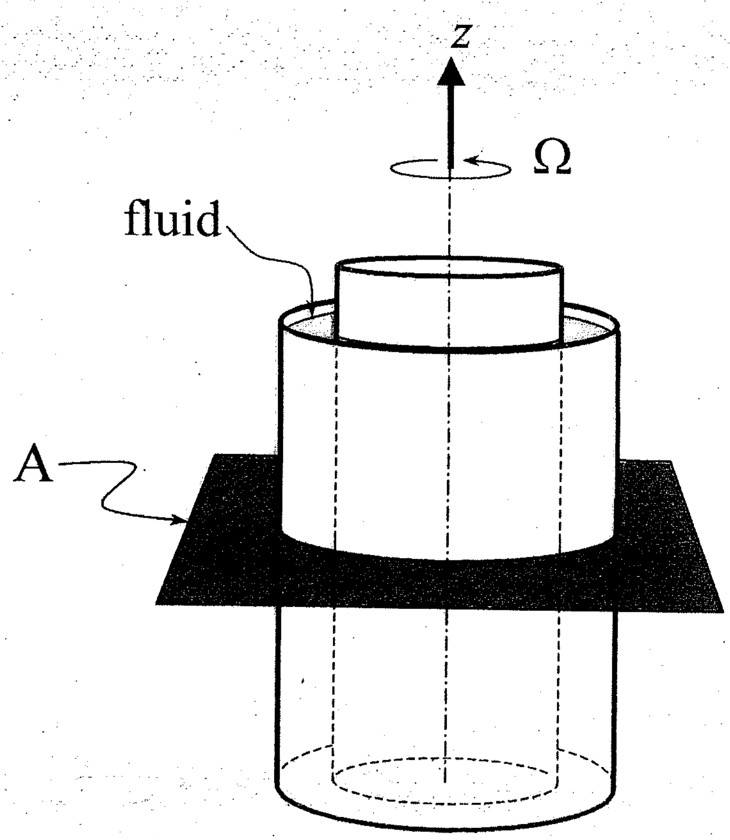
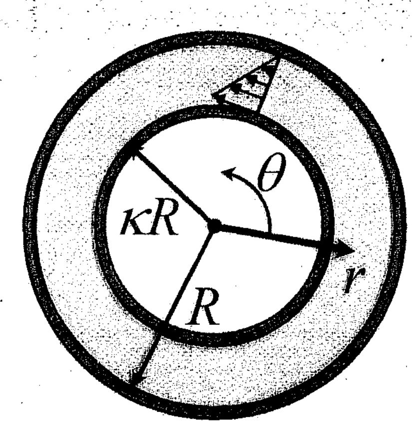
Example: Can the equation of motion predict rod climbing for typical values of N_1, N_2?



$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

cross-section A:



What is $\dfrac{d\Pi_{zz}}{dr}$?

Bird et al. p64

Soln

$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady state}} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \underbrace{\cancel{\rho\underline{g}}}_{\text{neglect } \underline{g}}$$

$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z} \qquad \nabla p = \begin{pmatrix} \frac{\partial p}{\partial r} \\ 0 \\ 0 \end{pmatrix}$$

← θ-symmetry

no z-dir variation

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}\tau_{\theta r}} + \cancel{\frac{\partial}{\partial z}\tau_{zr}} - \frac{\tau_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}\tau_{\theta\theta}} + \cancel{\frac{\partial}{\partial z}\tau_{z\theta}} + \cancel{\frac{\tau_{\theta r}-\tau_{r\theta}}{r}} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\cancel{\frac{\partial}{\partial \theta}\tau_{\theta z}} + \cancel{\frac{\partial}{\partial z}}(\tau_{zz}) \end{pmatrix}_{r\theta z}$$

(a vector)

θ symmetry

deep cylinder $\frac{\partial}{\partial z} = 0$

$\underline{\underline{\tau}}$ is symmetric

$$\nabla \cdot \underline{\underline{\tau}} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rr}\right) \cancel{\frac{\tau_{\theta\theta}}{r}} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\theta}\right) \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) \end{pmatrix}_{r\theta z}$$

with these assumptions:

$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

(θ-symmetry)

$$\underline{v} \cdot \nabla \underline{v} = \begin{pmatrix} -\frac{v_\theta^2}{r} \\ 0 \\ 0 \end{pmatrix}_{r\theta z}$$

EOM:

$$\begin{pmatrix} -\dfrac{\rho v_\theta^2}{r} \\ 0 \\ 0 \end{pmatrix}_{r\theta z} = \begin{pmatrix} -\dfrac{\partial P}{\partial r} \\ 0 \\ 0 \end{pmatrix}_{r\theta z} - \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rr}) - \dfrac{\tau_{\theta\theta}}{r} \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\tau_{r\theta} r^2) \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\tau_{rz}) \end{pmatrix}_{r\theta z}$$

Now, examine r-component of EOM for influence of N_1, N_2 on $P(r)$

$$-\frac{\rho v_\theta^2}{r} = -\frac{\partial P}{\partial r} - \frac{1}{r}\left(r\frac{\partial \tau_{rr}}{\partial r} + \tau_{rr}\right) + \frac{\tau_{\theta\theta}}{r}$$

$$N_1 = \tau_{11} - \tau_{22} = \tau_{\theta\theta} - \tau_{rr}$$

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial P}{\partial r} - \frac{\partial \tau_{rr}}{\partial r} \overset{+\frac{\partial \tau_{zz}}{\partial r}}{} \underbrace{- \frac{\tau_{rr}}{r} + \frac{\tau_{\theta\theta}}{r}}_{+\frac{N_1}{r}} + \frac{\partial \tau_{zz}}{\partial r}$$

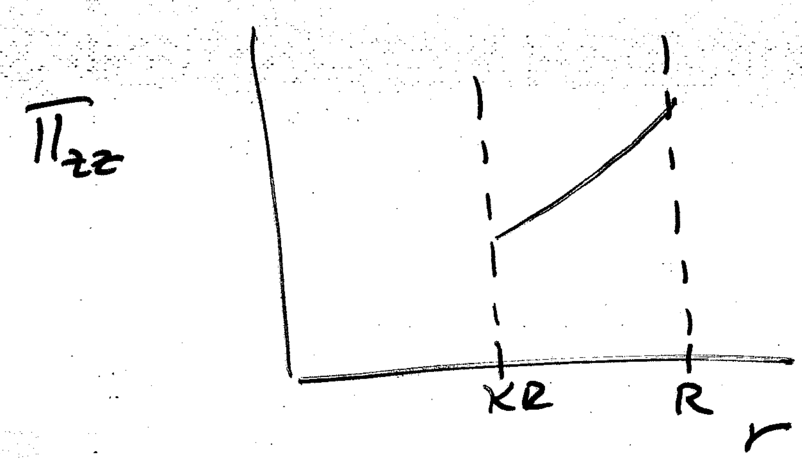
$$N_2 = \tau_{22} - \tau_{33} = \tau_{rr} - \tau_{zz}$$

$$\frac{\partial N_2}{\partial r} = \frac{\partial \tau_{rr}}{\partial r} - \frac{\partial \tau_{zz}}{\partial r}$$

$$-\rho\frac{v_\theta^2}{r} = -\frac{\partial}{\partial r}\overbrace{(P + \tau_{zz})}^{\Pi_{zz}} - \frac{\partial N_2}{\partial r} + \frac{N_1}{r}$$

$$\boxed{\frac{\partial \Pi_{zz}}{\partial r} = \rho\frac{v_\theta^2}{r} - \frac{\partial N_2}{\partial r} + \frac{N_1}{r}}$$

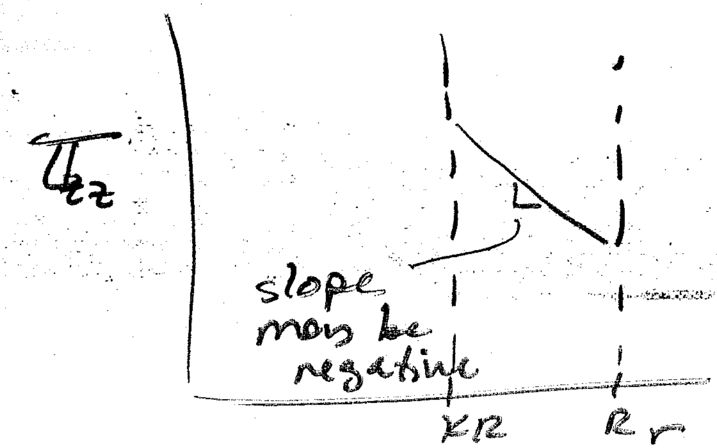
Π_{zz} = total stress on a z-surface acting in z-dir



For Newtonian FLUID

$N_1 = N_2 = 0$

$$\frac{\partial \Pi_{zz}}{\partial r} = \rho \frac{v_\theta^2}{r} > 0$$



FOR A POLYMERIC FLUID

$\tau_{11} - \tau_{22} = N_1 < 0$

$\tau_{22} - \tau_{33} = N_2 > 0$ (small)

If N_1 is sufficiently negative, the