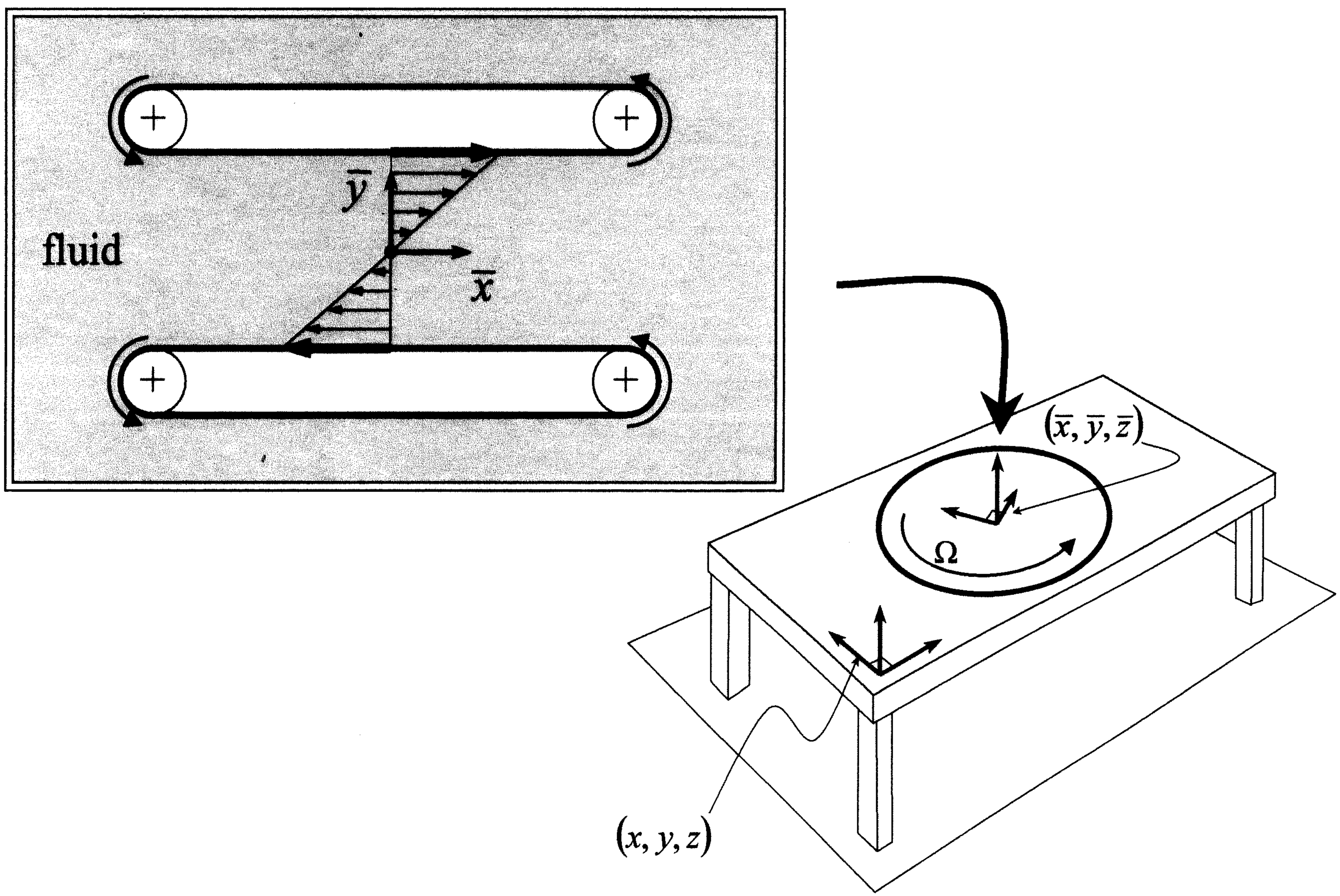
Shear flow in a rotating frame of reference



Shear flow from the different viewpoints of two observers:

1. one rotating with flow
2. one stationary

Both should give the same predictions for viscosity

$\bar{x}, \bar{y}, \bar{z}$ rotating, shear coordinate system

x, y, z stationary, non shear coord system (except when $t = 0$)

$\boxed{\text{Strategy: calculate } \eta \text{ predicted by GLVE model using both points of view}}$

PART I: (easy) Calc η in $\bar{x},\bar{y},\bar{z}$ system

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 \bar{y} \\ 0 \\ 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} \qquad \underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\gamma}_0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}}$$

Now calculate $\underline{\underline{\tau}}$ using GLVE:

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} G(t-t')\,\underline{\underline{\dot{\gamma}}}(t')\,dt'$$

$$= -\int_{-\infty}^{t} G(t-t') \begin{pmatrix} 0 & \dot{\gamma}_0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\bar{x}\bar{y}\bar{z}} dt'$$

defn of material function η:

$$\eta \equiv \frac{-\tau_{21}}{\dot\gamma_0} = \frac{1}{\cancel{\dot\gamma_0}} \int_{-\infty}^{t} G(t-t') \cancel{\dot\gamma_0}\, dt'$$

$s = t - t'$
$ds = -dt'$
$t' = -\infty \quad s = \infty$
$t' = t \quad s = 0$

$$\boxed{\eta = \int_0^\infty G(s)\, ds}$$

This is what we have always gotten in this calculation.

Now, carry out the same calculation in the stationary coord system .... (HARDER)

What is the velocity of fluid particles written with respect to a stationary observer?

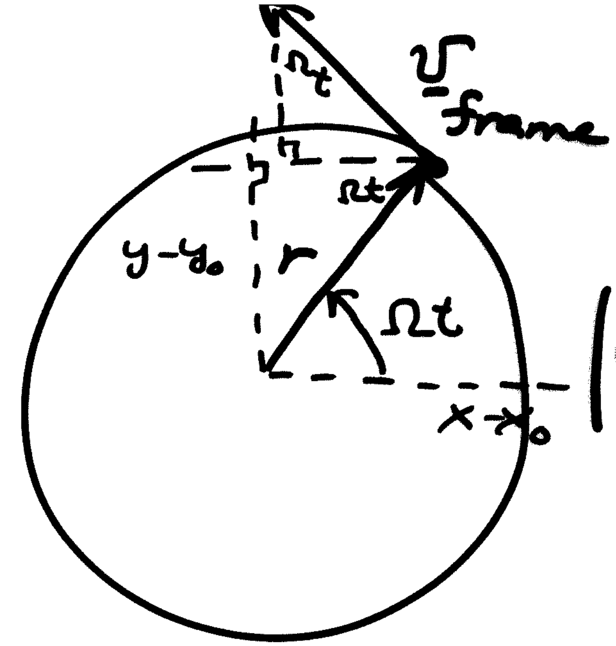
$$\underline{v}_{\substack{\text{wrt}\\ \text{stationary}\\ \text{frame}}} = \underline{v}_{\substack{\text{wrt}\\ \text{rotating}\\ \text{frame}}} + \underline{v}_{\text{frame}}$$

$$= \dot{\gamma}_0 \bar{y}\, \hat{e}_{\bar{x}} + \underline{v}_{\text{frame}}$$

need to write these wrt stationary frame

need to figure out what this is

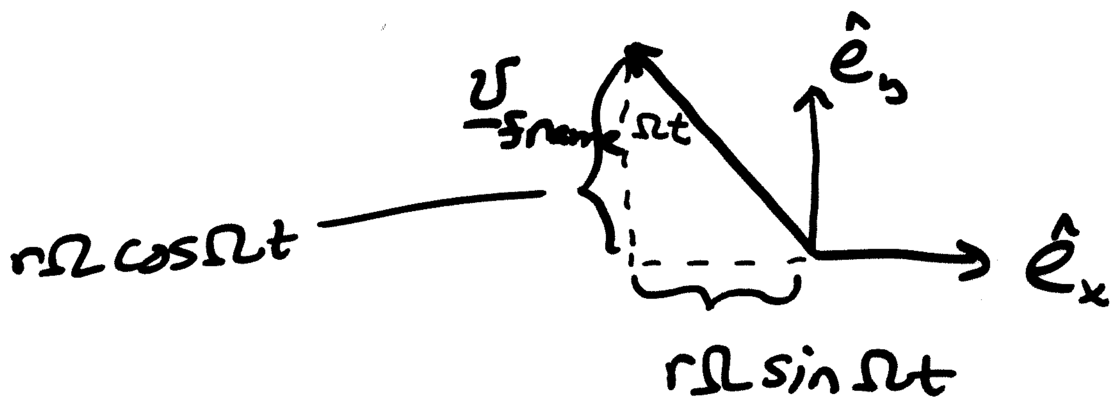
What is $\underline{v}_{frame}$?



$$|\underline{v}_{frame}| = r\Omega$$

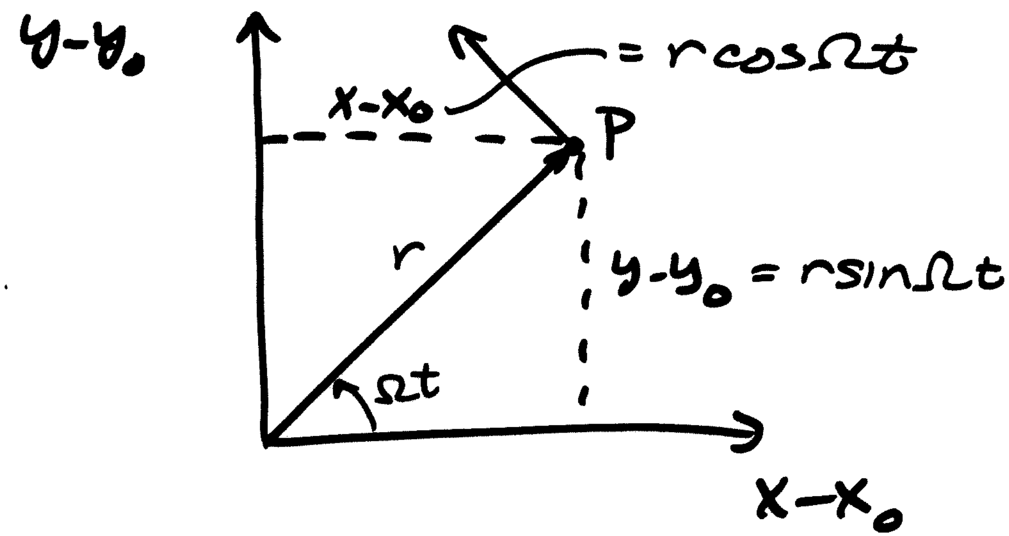
$\uparrow$ angular velocity

The velocity of any point on the turntable written in the stationary frame is $r\Omega$ pointing in a tangent direction.



$$\underline{v}_{frame} = -r\Omega\sin\Omega t\,\hat{e}_x + r\Omega\cos\Omega t\,\hat{e}_y$$

Point P:
$(x-x_0, y-y_0)$



$$\underline{v}_{frame} = -\Omega\underbrace{(r\sin\Omega t)}_{y-y_0}\,\hat{e}_x + \Omega\underbrace{(r\cos\Omega t)}_{x-x_0}\,\hat{e}_y$$

$$\boxed{\underline{v}_{frame} = -\Omega(y-y_0)\,\hat{e}_x + \Omega(x-x_0)\,\hat{e}_y}$$

The next step is the coordinate transformation of the fluid velocity....

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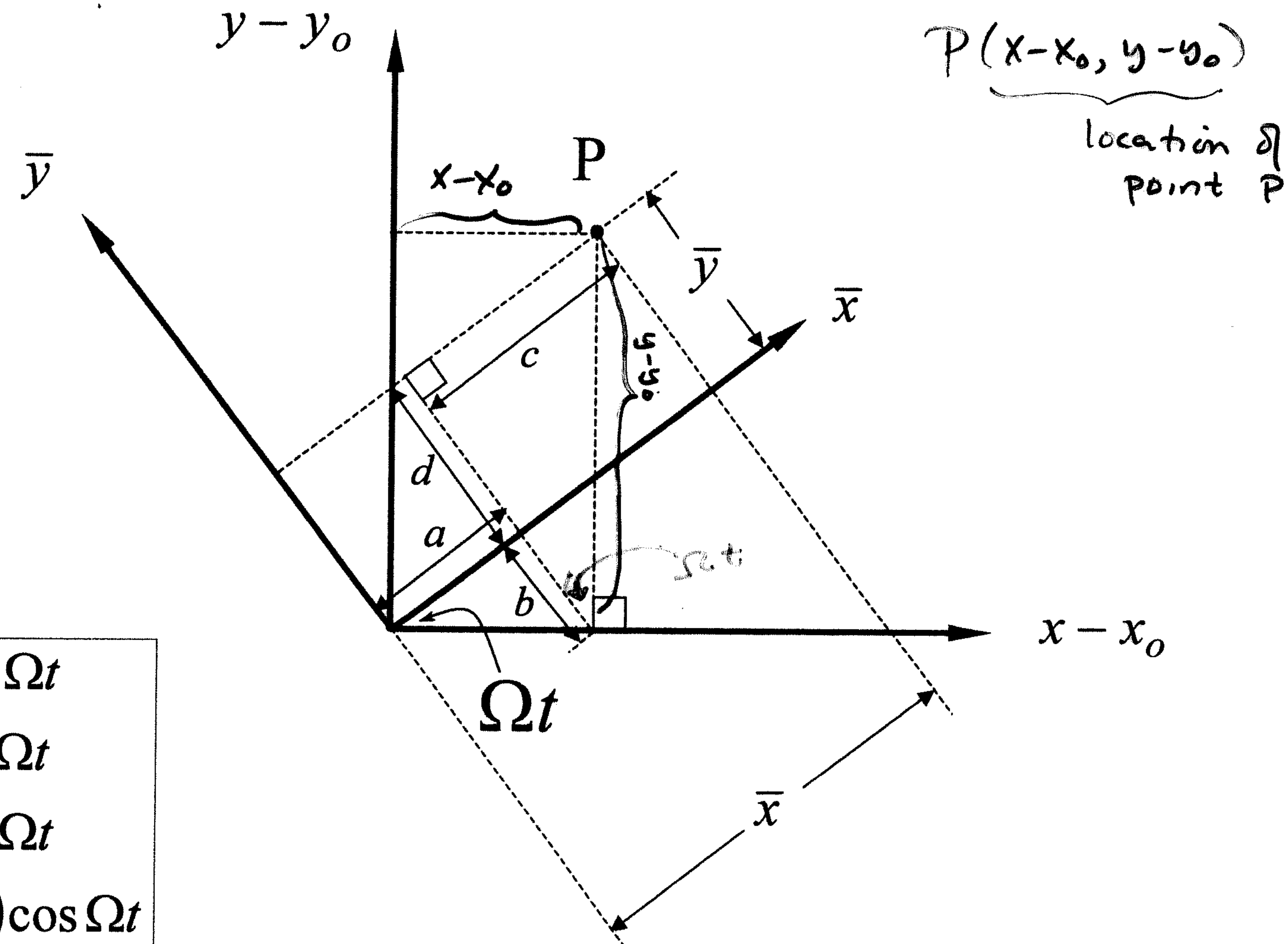


$$a = (x - x_o)\cos\Omega t$$
$$b = (x - x_o)\sin\Omega t$$
$$c = (y - y_o)\sin\Omega t$$
$$d + b = (y - y_o)\cos\Omega t$$

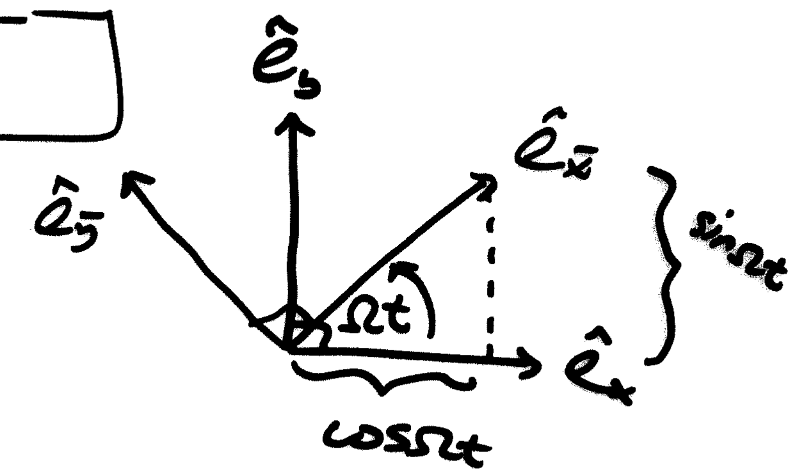
$\dot{\gamma}_0 \bar{y} \hat{e}_{\bar{x}}$

what is $\hat{e}_{\bar{x}}$ in terms of $\hat{e}_x, \hat{e}_y$?

what is $\bar{y}$ in terms of $x-x_0, y-y_0$?

$$\bar{y} = d = (y-y_0)\cos\Omega t - \underbrace{b}_{(x-x_0)\sin\Omega t}$$

$$\boxed{\bar{y} = (y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t}$$



$$\boxed{\hat{e}_{\bar{x}} = \cos\Omega t \, \hat{e}_x + \sin\Omega t \, \hat{e}_y}$$

NOW, ASSEMBLE ....

$$\underline{v}_{\substack{\text{wrt}\\ \text{stationary}\\ \text{frame}}} = \dot{\gamma}_0 \bar{y}\, \hat{e}_x + \underline{v}_{\text{frame}}$$

$$= \dot{\gamma}_0 \left[(y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t \right] \left[\cos\Omega t\, \hat{e}_x + \sin\Omega t\, \hat{e}_y \right]$$

$$+ \left[-\Omega(y-y_0)\hat{e}_x + \Omega(x-x_0)\hat{e}_y \right]$$

$$= \begin{pmatrix} \dot{\gamma}_0 \cos\Omega t \left[(y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t \right] \\ -\Omega(y-y_0) \\ \\ \dot{\gamma} \sin\Omega t \left[(y-y_0)\cos\Omega t - (x-x_0)\sin\Omega t \right] \\ +\Omega(x-x_0) \\ \\ 0 \end{pmatrix}_{xyz}$$

We need $\underline{\underline{\dot{\gamma}}}$ for the constitutive eqn, from which we'll calculate τ_{21} and η.

$$\underline{\underline{\dot{\gamma}}} = \nabla\underline{v} + (\nabla\underline{v})^T$$

$$\nabla\underline{v} = \begin{pmatrix} -\sin\Omega t\,\dot{\gamma}_0\cos\Omega t & -\sin\Omega t\,\dot{\gamma}_0\sin\Omega t + \Omega & 0 \\ \dot{\gamma}_0\cos^2\Omega t - \Omega & \dot{\gamma}_0\sin\Omega t\cos\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} -\overbrace{2\sin\Omega t\cos\Omega t}^{\sin 2\Omega t}\,\dot{\gamma}_0 & \dot{\gamma}_0\overbrace{(\cos^2\Omega t - \sin^2\Omega t)}^{\cos 2\Omega t} & 0 \\ \dot{\gamma}_0 \cos^2\Omega t - \sin^2\Omega t & 2\sin\Omega t\cos\Omega t\,\dot{\gamma}_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} -(\sin 2\Omega t)\,\dot{\gamma}_0 & \dot{\gamma}_0 \cos 2\Omega t & 0 \\ \dot{\gamma}_0 \cos 2\Omega t & \dot{\gamma}_0 \sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Now, we predict η: $\eta = \dfrac{-\tau_{21}}{\dot{\gamma}_0}$

This definition is tied to the shear coordinate system 123, but xyz is not a shear coord system <u>except when</u> $\underline{t=0}$.

$$\tau_{yx}(t) = -\int_{-\infty}^{t} G(t-t')\,\dot{\gamma}_{yx}(t')\,dt'$$

$s = t - t'$

$ds = -dt'$

$t' = -\infty \quad s = \infty$

$t' = t \quad s = 0$

$$\tau_{yx}(t) = -\int_0^{\infty} G(s)\,\dot{\gamma}_{yx}(t-s)\,ds$$

$$= -\int_0^{\infty} \dot{\gamma}_0\, G(s) \cos(2\Omega(t-s))\,ds$$

We can predict η only when $t=0$

at $t=0$ $\quad \tau_{yx} = \tau_{xy}$

$$\eta = \frac{-\tau_{xy}}{\dot{\gamma}_0} = \frac{\cancel{\dot{\gamma}_0} \int_0^{\infty} G(s)\cos(2\Omega(\overset{0}{t}-s))\,ds}{\cancel{\dot{\gamma}_0}}$$

$$\eta = \int_0^{\infty} G(s)\cos(-2\Omega s)\,ds$$

$$\boxed{\eta = \int_0^{\infty} G(s)\cos 2\Omega s\,ds}$$

The predicted value of η depends on Ω!!

Rotating Frame: $\eta = \int_0^\infty G(s)\,ds$

Stationary Frame: $\eta = \int_0^\infty G(s)\cos 2\Omega s\,ds$ ✗

★ PROBLEMS:

① should be the same
② should <u>not</u> depend on Ω

This is incorrect.
This shows that the GLVE eqn is not frame-invariant.

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