9.57 Show that the stress tensor in rubber elasticity theory is as given below:

$$\underline{\underline{\tau}} = -\nu k T \lambda_i^2 \hat{e}_i \hat{e}_i$$

$$\underline{\underline{\tau}} = -\frac{3kT\nu}{Na^2} \langle \underline{R}\,\underline{R} \rangle \qquad (\text{EQN } 9.362)$$

$$= -\frac{3kT\nu}{Na^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{R}\,\underline{R}\,\Psi(\underline{R})\, dR_1\, dR_2\, dR_3$$

$$\frac{\underline{\underline{\tau}}\, Na^2}{-3kT\nu} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\underline{R}\,\underline{R}}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 e^{-\beta^2\left(\left(\frac{R_1}{\lambda_1}\right)^2 + \left(\frac{R_2}{\lambda_2}\right)^2 + \left(\frac{R_3}{\lambda_3}\right)^2\right)} dR_1\, dR_2\, dR_3$$

$$\sum_{i=1}^{3} \sum_{k=1}^{3} R_i \hat{e}_i R_k \hat{e}_k$$

$$\sum_{i=1}^{3} \sum_{k=1}^{3} R_i R_k \hat{e}_i \hat{e}_k$$

THERE ARE NINE INTEGRATIONS TO PERFORM.

(1770)

FOR $i = k$ $\qquad p \neq i \qquad$ $m \neq i$, $m \neq p \qquad$ $i = 1,2,3$

$$\left(\frac{\sqrt{\pi}}{\beta}\right)^3 \frac{\tau_{ii} N a^2}{-3kTU} = \int_{-\infty}^{\infty} e^{-\beta^2\left(\frac{R_p}{\lambda_p}\right)^2} dR_p \int_{-\infty}^{\infty} e^{-\beta^2\left(\frac{R_m}{\lambda_m}\right)^2} dR_m \int_{-\infty}^{\infty} R_i^2 e^{-\beta^2\left(\frac{R_i}{\lambda_i}\right)^2} dR_i$$

THE FIRST 2 INTEGRALS ARE OF THIS TYPE:

$$\int_{-\infty}^{\infty} e^{-\beta^2\left(\frac{R_p}{\lambda_p}\right)^2} dR_p$$

$$= \frac{\lambda_p 2}{\beta} \underbrace{\int_0^{\infty} e^{-\beta^2 \frac{R_p^2}{\lambda_p^2}} \frac{\beta}{\lambda_p} dR_p}_{\frac{\sqrt{\pi}}{2} \operatorname{erf}(\infty) = \frac{\sqrt{\pi}}{2}}$$

$$= \frac{\lambda_p}{\beta} \sqrt{\pi}$$

THE THIRD INTEGRAL IS

$$\int_{-\infty}^{\infty} R_i^2 e^{-\beta^2\left(\frac{R_i}{\lambda_i}\right)^2} dR_i$$

SINCE THE INTEGRAND IS AN EVEN FN,

$$= 2 \int_0^{\infty} R_i^2 e^{-\beta^2 (R_i/\lambda_i)^2} dR_i \qquad (77)$$

We'll use Mathcad to evaluate the integral. We begin with the function below.

$$R^2 \cdot \exp\left(\frac{-\beta^2 \cdot R^2}{\lambda^2}\right)$$

Integrating on R yields:

$$\frac{-1}{\left(2 \cdot \beta^2\right)} \cdot \lambda^2 \cdot R \cdot \exp\left(-\beta^2 \cdot \frac{R^2}{\lambda^2}\right) + \frac{1}{\left(4 \cdot \beta^3\right)} \cdot \lambda^3 \cdot \sqrt{\pi} \cdot \operatorname{erf}\left(\frac{\beta}{\lambda} \cdot R\right)$$

Evaluating this at infinity we get

$$\frac{1}{\left(4 \cdot \beta^3\right)} \cdot \lambda^3 \cdot \sqrt{\pi}$$

Evaluating this at zero we get zero. Thus the integral is equal to

$$\frac{2}{\left(4 \cdot \beta^3\right)} \cdot \lambda^3 \cdot \sqrt{\pi}$$

COMBINING THE EXPRESSIONS, WE OBTAIN

$$\left(\cancel{\frac{\sqrt{\pi}}{\beta}}\right)^3 \frac{\tau_{ii}\, Na^2}{-3kTV} = \underset{p \neq i}{\lambda_p} \left(\cancel{\frac{\sqrt{\pi}}{\beta}}\right) \underset{\substack{m \neq i \\ m \neq p}}{\lambda_m} \left(\cancel{\frac{\sqrt{\pi}}{\beta}}\right) \frac{\cancel{\sqrt{\pi}}\, \lambda_i^3}{2\cancel{\beta^3}} \hat{e}_i \hat{e}_i$$

$$\beta^2 = \frac{3}{2Na^2}$$

$$\tau_{ii} = -\overbrace{\frac{3}{2} \frac{1}{Na^2}}^{\beta^2} \frac{1}{\cancel{\beta^2}} VkT\, \lambda_i^3 \lambda_p \lambda_m \hat{e}_i \hat{e}_i$$

(7.72)

DUE TO INCOMPRESSIBILITY,

$$\lambda_1 \lambda_2 \lambda_3 = 1$$

$$\lambda_i \lambda_p \lambda_m = 1 \qquad \begin{matrix} p \neq i \\ m \neq i \\ m \neq p \end{matrix}$$

$$\Rightarrow \lambda_p \lambda_m = \frac{1}{\lambda_i}$$

WE OBTAIN FOR THE DIAGONAL COMPONENTS:

$$\boxed{\tau_{ii} = -\nu k T \lambda_i^2 \, \hat{e}_i \hat{e}_i}$$

NOW WE MUST CALCULATE τ_{ik} FOR $i \neq k$, THE OFF-DIAGONAL COMPONENTS.

FOR $i \neq k$ $\quad j \neq i$
$i = 1, 2, 3$ $\quad j \neq k$
$k = 1, 2, 3$

$$\left(\frac{\sqrt{\pi}}{\beta}\right)^3 \frac{\tau_{ik} N a^2}{-3kTV} = \int_{-\infty}^{\infty} e^{-\beta^2\left(\frac{R_j}{\lambda_j}\right)^2} dR_j \int_{-\infty}^{\infty} R_i \, e^{-\beta^2\left(\frac{R_i}{\lambda_i}\right)^2} dR_i$$

$$* \int_{-\infty}^{\infty} R_k \, e^{-\beta^2\left(\frac{R_k}{\lambda_k}\right)^2} dR_k$$

THE FIRST INTEGRAL IS JUST $\frac{\lambda_j}{\beta}\sqrt{\pi}$ AS BEFORE

THE SECOND TWO INTEGRALS ARE

$$\int_{-\infty}^{\infty} \underbrace{R_i \, e^{-\beta^2\left(\frac{R_i}{\lambda_i}\right)^2}}_{} dR_i$$

THIS IS NOT AN EVEN
FUNCTION. ∴ WE DO NOT
CHANGE LIMITS.

$$= \int_{-\infty}^{\infty} \frac{\lambda_i^2}{-\beta^2} \frac{1}{2} \, e^{-\beta^2\left(\frac{R_i^2}{\lambda_i^2}\right)} \left(dR_i \frac{-\beta^2}{\lambda_i^2} 2R_i\right)$$

(774)

$$\int_{-\infty}^{\infty} R_i \, e^{-\beta^2 \left(\frac{R_i}{\lambda^2}\right)^2} dR_i$$

$$\frac{\lambda_i^2}{-\beta^2} \frac{1}{2} e^{-\beta^2 \frac{R_i^2}{\lambda_i^2}} \Big|_{-\infty}^{\infty} = 0$$

THUS $\tau_{ik} = 0$ FOR $i \neq k$.

THE FINAL RESULT FOR $\underline{\underline{\tau}}$ IS THEN

$$\boxed{\underline{\underline{\tau}} = -\nu k T \lambda_i^2 \, \hat{e}_i \hat{e}_i}$$