What we've learned so far on Fluid Mechanics?

**Why?**
- trial + error is not getting us far in high tech engineering
- need mathematical models

**Inventory**
- MEB
- $P_{net} = P_{top} + gsh$ - fluid statics
- experience w/ fluids
- calculus, vectors, coordinate systems
**NEW to us in this course:**

- **Viscosity** \( \tau_{yx} = \mu \frac{dV_x}{dy} \)

- **Stress tensor** \( \mathbf{T} = -p \mathbb{I} + \mathbf{\tau} \)

\[ \mathbf{T} = \mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

- **Field - continuum**

I choose a position; the field variable tells me the value

\[ \phi(x, y, z) \]

\[ \mathbf{T}(x, y, z) \]
→ Force on surface within a fluid

\[ F = \iint_S (\hat{n} \cdot \vec{F}) \, ds \]

→ Flow rate through surface

\[ Q = \iiint_S \hat{n} \cdot \vec{v} \, ds \]

→ Mass flow rate

\[ m = \rho Q \]
→ Control volume - to organize balancing mass, momentum, energy in a continuum

- momentum bal. on a c.v.
  \[ \sum F = \frac{dP}{dt} + \int_{S} n \cdot v \, ds \]
  (Reynolds Xpart Thm)

- contact moment form
- pressure
- viscous
- gravity
Navier-Stokes equations

\[ P \left( \frac{\partial V}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{V} \right) = -\nabla P + \mu \nabla^2 \mathbf{V} + \mathbf{f} \]

- **momentum** balance at a point in a continuum
- **pressure**
- **viscous** force
- **gravity**

- **rate of change**
- **convective** term
- **momentum**

sum of forces
4. (25 points) For the steady flow of water through two long, wide plates (see figure), calculate the velocity profile. The plates are tilted from the horizontal at an angle $\alpha$. Show all your steps; no credit will be given for the right answer without the accompanying calculations and justification. Use the coordinate system given; note $x_1=x$, $x_2=y$ and $x_3=z$. You may neglect the $x_2$-component of gravity, but do not neglect the $x_1$-component of gravity.

\[ \mathbf{u} = \begin{pmatrix} u_1 \\ 0 \\ 0 \end{pmatrix} \]

\[ x_1 = x, \quad x_2 = y, \quad x_3 = z \]

steady \[ \mathbf{u} = \mathbf{u}, \mathbf{e} \]
The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

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Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 \rho v_r)}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} - \frac{1}{r} \frac{\partial (r \tau_{xx})}{\partial r} - \frac{\partial \tau_{x,y}}{\partial y} - \frac{\partial \tau_{x,z}}{\partial z} + \rho g_x$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{\partial \theta} - \frac{v_r^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} - \frac{1}{r} \frac{\partial (r \tau_{r,r})}{\partial \theta} + \frac{1}{r} \frac{\partial \tau_{r,\theta}}{\partial \theta} + \frac{\partial \tau_{r,z}}{\partial z} + \rho g_r$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{\partial \theta} + \frac{v_r^2}{r} \frac{\partial v_r}{\partial \phi} \right) = -\frac{\partial P}{\partial r} - \frac{1}{r^2} \frac{\partial (r^2 \tau_{r,r})}{\partial \theta} + \frac{1}{r} \frac{\partial \tau_{r,\theta}}{\partial \theta} + \frac{\partial \tau_{r,\phi}}{\partial \phi} + \frac{\tau_{\phi,\phi}}{r} + \frac{\cot \theta \tau_{\phi,\phi}}{r} + \rho g_{\phi}$$

Conclusion (or end)
Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

\[
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x
\]

\[
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y
\]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z
\]

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

\[
\rho \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{v_r}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r
\]

\[
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial r^2} + \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{v_\theta^2 / \sin \theta + v_z^2 / \sin \theta}{r} \right) + \rho g_\theta
\]

\[
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{1}{r \sin \theta} \frac{\partial P}{\partial \phi} + \mu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_z}{\partial \theta} + \frac{2 \cos \theta \frac{\partial v_z}{\partial \phi}}{r} \right) + \rho g_\phi
\]

where, in these equations, \( \nabla = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \).

Advice: LEAVE PRESSURE until LAST moment.

Usually, the momentum bal. answers the pressure questions by itself.

gravity vector in our chosen coordinate system:

\[
g = -g \sin \theta, -g \cos \theta
\]

Method:
- Sketch vector, sketch coord sys
- Extend coord system axes
- Find 2 vectors parallel to coord axes that add up to \( g \)
$z$-component
\[
\frac{\partial P}{\partial x_3} = \frac{\partial P}{\partial z} = 0 \quad \Rightarrow \quad z = x_3
\]

$y$-component
\[
\frac{\partial P}{\partial y} = -\rho g \cos \alpha \quad \Rightarrow \quad y = x_2
\]

$x$-component
\[
\frac{\partial P}{\partial x_1} = \mu \frac{d^2 V_1}{dx_1^2} - \rho g \sin \alpha
\]

$P = P(x_1)$ only
$V_1 = V_1(x_2)$ only

allows us to separate the two variables