Equations Summary from Inside Cover of Morrison, 2013

Mechanical Energy Balance

$$\frac{\Delta p}{\rho} + \frac{\Delta\langle v\rangle^2}{2\alpha} + g\Delta z + F_{friction} = -\frac{W_{s,by\ fluid}}{m}$$

$$\begin{cases}\alpha_{laminar} = 0.5 \\ \alpha_{turbulent} \approx 1\end{cases}$$

$$F_{friction} = \left[4f\frac{L}{D} + \sum_{fittings_i} n_i K_{f,i}\right]\frac{\langle v\rangle^2}{2}$$

Fanning Friction Factor (pipe flow)

$$f = \frac{F_{drag}}{\frac{1}{2}\rho\langle v\rangle^2\,(2\pi RL)} = \frac{\Delta p D}{2L\rho\langle v\rangle^2}$$

Note this is <u>correct</u>; there is an error on the inside cover

Drag Coefficient (sphere drop)

$$C_D = \frac{F_{drag}}{\frac{1}{2}\rho v_\infty^2\,(\pi R^2)} = \frac{4gD(\rho_{body} - \rho)}{3\rho v_\infty^2}$$

Momentum balance on a CV (Reynolds transport theorem)

$$\frac{d\underline{\mathbf{P}}}{dt} + \iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{on\ CV}\underline{f}$$

Hydrostatic Pressure

$$p_{bottom} = p_{top} + \rho g h$$

Hagen-Poiseuille Equation (steady, laminar tube flow, incompressible)

$$Q = \frac{\pi(p_0 - p_L)R^4}{8\mu L}$$

Prandtl Equation (steady, turbulent tube flow)

$$\frac{1}{\sqrt{f}} = -4.0\log\left(\frac{4.67}{Re\sqrt{f}}\right) + 2.28$$

Stokes-Einstein-Sutherland Equation (steady, slow flow around a sphere)

$$F_{drag} = 6\pi R\mu v_\infty$$

Macroscopic Momentum Balance on a CV

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos(\theta)\langle v\rangle^2}{\beta}\hat{v}\right]\Bigg\|_{A_i} = \sum_{i=1}^{\#streams}[-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$$\begin{cases}\beta_{laminar} = 0.75 \\ \beta_{turbulent} \approx 1\end{cases}$$

Navier-Stokes equation (microscopic momentum balance, incompressible, Newtonian fluids)

$$\rho\left(\frac{\partial\underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Continuity equation (microscopic mass balance, incompressible fluids)

$$\nabla\cdot\underline{v} = 0$$

Total stress tensor $\quad \underline{\tilde{\underline{\Pi}}} = -p\underline{I} + \underline{\tilde{\tau}}$

$$\begin{pmatrix} \tilde{\Pi}_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} \\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} \\ \tilde{\Pi}_{31} & \tilde{\Pi}_{32} & \tilde{\Pi}_{33} \end{pmatrix}_{123} = \begin{pmatrix} \tilde{\tau}_{11} - p & \tilde{\tau}_{12} & \tilde{\tau}_{13} \\ \tilde{\tau}_{21} & \tilde{\tau}_{22} - p & \tilde{\tau}_{23} \\ \tilde{\tau}_{31} & \tilde{\tau}_{32} & \tilde{\tau}_{33} - p \end{pmatrix}_{123}$$

Dynamic pressure $\quad \mathcal{P} \equiv p + \rho g h$

Newtonian constitutive equation
$$\underline{\tilde{\tau}} = \mu \left(\nabla \underline{v} + (\nabla \underline{v})^T \right)$$

$$= \mu \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

Total molecular fluid force on a finite surface $\mathcal{S}$
$$\underline{\mathcal{F}} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{\tilde{\underline{\Pi}}} \right]_{\text{at surface}} dS$$

Stationary fluid $\quad \left[\hat{n} \cdot \underline{\tilde{\underline{\Pi}}} \right] = -p\hat{n}$

Moving fluid $\quad \left[\hat{n} \cdot \underline{\tilde{\underline{\Pi}}} \right] = -p\hat{n} + \hat{n} \cdot \underline{\tilde{\tau}}$

Total fluid torque on a finite surface $\mathcal{S}$
$$\underline{\mathcal{T}} = \iint_{\mathcal{S}} \left[\underline{R} \times \left(\hat{n} \cdot \underline{\tilde{\underline{\Pi}}} \right) \right]_{\text{at surface}} dS$$

Total flow rate out through a finite surface $\mathcal{S}$
$$Q = \dot{V} = \iint_{\mathcal{S}} \left[\hat{n} \cdot \underline{v} \right]_{\text{at surface}} dS$$

Average velocity across a finite surface $\mathcal{S}$
$$\langle v \rangle = \frac{Q}{\mathcal{S}}$$

Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta dr d\theta d\phi$

Coordinate system	coordinates	basis vectors
spherical	$x = r\sin\theta\cos\phi$	$\hat{e}_r = (\sin\theta\cos\phi\hat{e}_x) + (\sin\theta\sin\phi\hat{e}_y) + \cos\theta\hat{e}_z$
	$y = r\sin\theta\sin\phi$	$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$
	$z = r\cos\theta$	$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\hat{e}_y$
cylindrical	$x = r\cos\theta$	$\hat{e}_r = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y$
	$y = r\sin\theta$	$\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\hat{e}_y$
	$z = z$	$\hat{e}_z = \hat{e}_z$

Divergence Theorem
$$\iint_{\mathcal{S}} \hat{n} \cdot \underline{F}\ dS = \iiint_{\mathcal{V}} \nabla \cdot \underline{F}\ dV$$

Stokes Theorem
$$\oint_{\mathcal{C}} \hat{t} \cdot \underline{F}\ dl = \iint_{\mathcal{S}} \hat{n} \cdot (\nabla \times \underline{F})\ dS$$

Vector identities:
$$\nabla \cdot \nabla \times \underline{F} = 0 \quad \text{(Divergence of curl} = 0)$$
$$\nabla \times \nabla f = 0 \quad \text{(Curl of gradient} = 0)$$
$$\nabla (fg) = f\nabla g + g\nabla f$$
$$\underline{F} \cdot \nabla \underline{F} = \frac{1}{2}\nabla (\underline{F}^2) - \underline{F} \times (\nabla \times \underline{F})$$
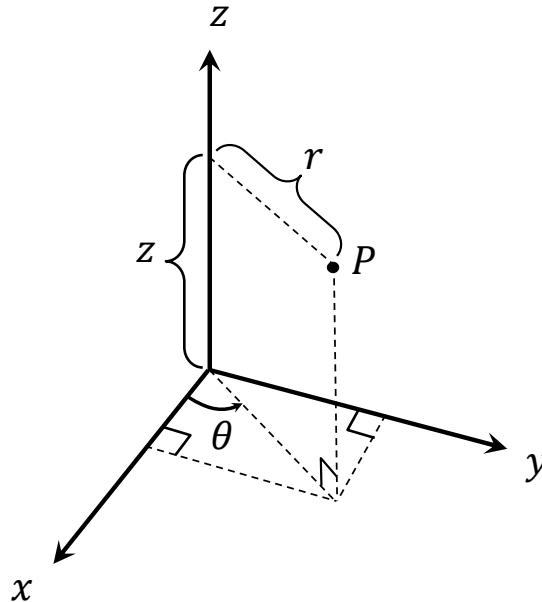$$\nabla \cdot (f\underline{F}) = f\nabla \cdot \underline{F} + \underline{F} \cdot \nabla f$$
$$\nabla \times \nabla \times \underline{F} = \nabla (\nabla \cdot \underline{F}) - \nabla^2 \underline{F}$$
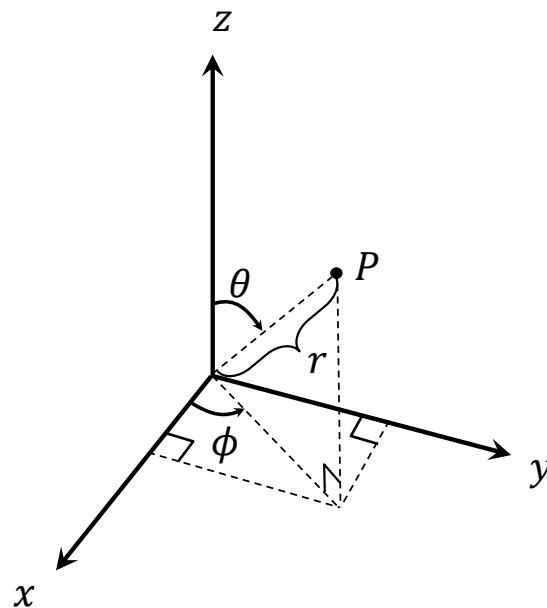$$\nabla \cdot (\underline{F} \times \underline{G}) = \underline{G} \cdot (\nabla \times \underline{F}) - \underline{F} \cdot (\nabla \times \underline{G})$$

The equations in F. A. Morrison, ***An Introduction to Fluid Mechanics*** (Cambridge, 2013) assume the following definitions of the cylindrical and spherical coordinate systems.

Cylindrical Coordinate System: Note that the θ-coordinate swings around the z-axis

Spherical Coordinate System: Note that the θ-coordinate swings down from the z-axis; this is different from its definition in the cylindrical system above.

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta r}}{\partial \theta} - \frac{\tilde{\tau}_{\theta\theta}}{r} + \frac{\partial \tilde{\tau}_{zr}}{\partial z} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta\theta}}{\partial \theta} + \frac{\partial \tilde{\tau}_{z\theta}}{\partial z} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{1}{r}\frac{\partial (r\tilde{\tau}_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tilde{\tau}_{\theta z}}{\partial \theta} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

Equation of Motion for an incompressible fluid, 3 components in spherical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right)$$
$$= -\frac{\partial P}{\partial r} + \left(\frac{1}{r^2}\frac{\partial (r^2\tilde{\tau}_{rr})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta r} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi r}}{\partial \phi} - \frac{\tilde{\tau}_{\theta\theta} + \tilde{\tau}_{\phi\phi}}{r} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r} \right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\theta})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta\theta} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi\theta}}{\partial \phi} + \frac{\tilde{\tau}_{\theta r} - \tilde{\tau}_{r\theta}}{r} - \frac{\tilde{\tau}_{\phi\phi} \cot\theta}{r} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta \cot\theta}{r} \right)$$
$$= -\frac{1}{r \sin\theta}\frac{\partial P}{\partial \phi} + \left(\frac{1}{r^3}\frac{\partial (r^3\tilde{\tau}_{r\phi})}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\tilde{\tau}_{\theta\phi} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \tilde{\tau}_{\phi\phi}}{\partial \phi} + \frac{\tilde{\tau}_{\phi r} - \tilde{\tau}_{r\phi}}{r} + \frac{\tilde{\tau}_{\phi\theta} \cot\theta}{r} \right) + \rho g_\phi$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in spherical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right)$$
$$= -\frac{\partial P}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right.$$
$$\left. - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2\cot\theta}{r}\right)$$
$$= -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}\right.$$
$$\left. + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\phi v_\theta\cot\theta}{r}\right)$$
$$= -\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} + \mu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi\sin\theta\right)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}\right.$$
$$\left. + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right) + \rho g_\phi$$

Note: the r-component of the Navier-Stokes equation in spherical coordinates may be simplified by adding $0 = \frac{2}{r}\nabla\cdot\underline{v}$ to the component shown above. This term is zero due to the continuity equation (mass conservation). See Bird et. al.

References:

1. R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena*, 2nd edition, Wiley: NY, 2002.

2. R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Fluids: Volume 1 Fluid Mechanics*, Wiley: NY, 1987.

FACTORS FOR UNIT CONVERSIONS

Quantity	Equivalent Values

Mass

$1 \text{ kg} = 1000 \text{ g} = 0.001 \text{ metric ton} = 2.20462 \text{ lb}_m = 35.27392 \text{ oz}$

$1 \text{ lb}_m = 16 \text{ oz} = 5 \times 10^{-4} \text{ ton} = 453.593 \text{ g} = 0.453593 \text{ kg}$

Length

$1 \text{ m} = 100 \text{ cm} = 1000 \text{ mm} = 10^6 \text{ microns } (\mu m) = 10^{10} \text{ angstroms (Å)}$

$= 39.37 \text{ in} = 3.2808 \text{ ft} = 1.0936 \text{ yd} = 0.0006214 \text{ mile}$

$1 \text{ ft} = 12 \text{ in.} = 1/3 \text{ yd} = 0.3048 \text{ m} = 30.48 \text{ cm}$

Volume

$1 \text{ m}^3 = 1000 \text{ liters} = 10^6 \text{ cm}^3 = 10^6 \text{ ml}$

$= 35.3145 \text{ ft}^3 = 220.83 \text{ imperial gallons} = 264.17 \text{ gal}$

$= 1056.68 \text{ qt}$

$1 \text{ ft}^3 = 1728 \text{ in}^3 = 7.4805 \text{ gal} = 0.028317 \text{ m}^3 = 28.317 \text{ liters}$

$= 28\ 317 \text{ cm}^3$

Force

$1 \text{ N} = 1 \text{ kg·m/s}^2 = 10^5 \text{ dynes} = 10^5 \text{ g·cm/s}^2 = 0.22481 \text{ lb}_f$

$1 \text{ lb}_f = 32.174 \text{ lb}_m\text{·ft/s}^2 = 4.4482 \text{ N} = 4.4482 \times 10^5 \text{ dynes}$

Pressure

$1 \text{ atm} = 1.01325 \times 10^5 \text{ N/m}^2 \text{ (Pa)} = 101.325 \text{ kPa} = 1.01325 \text{ bars}$

$= 1.01325 \times 10^6 \text{ dynes/cm}^2$

$= 760 \text{ mm Hg at } 0°\text{ C (torr)} = 10.333 \text{ m H}_2\text{O at } 4°\text{ C}$

$= 14.696 \text{ lb}_f/\text{in}^2 \text{ (psi)} = 33.9 \text{ ft H}_2\text{O at } 4°\text{C}$

$100 \text{ kPa} = 1 \text{ bar}$

Energy

$1 \text{ J} = 1 \text{ N·m} = 10^7 \text{ ergs} = 10^7 \text{ dyne·cm}$

$= 2.778 \times 10^{-7} \text{ kW·h} = 0.23901 \text{ cal}$

$= 0.7376 \text{ ft·lb}_f = 9.47817 \times 10^{-4} \text{ Btu}$

Power

$1 \text{ W} = 1 \text{ J/s} = 0.23885 \text{ cal/s} = 0.7376 \text{ ft·lb}_f/\text{s} = 9.47817 \times 10^{-4} \text{ Btu/s} = 3.4121 \text{ Btu/h}$

$= 1.341 \times 10^{-3} \text{ hp (horsepower)}$

Viscosity

$1 \text{ Pa·s} = 1 \text{ N·s/m}^2 = 1 \text{ kg/m·s}$

$= 10 \text{ poise} = 10 \text{ dynes·s/cm}^2 = 10 \text{ g/cm·s}$

$= 10^3 \text{ cp (centipoise)}$

$= 0.67197 \text{ lb}_m/\text{ft·s} = 2419.088 \text{ lb}_m/\text{ft·h}$

Density

$1 \text{ kg/m}^3 = 10^{-3} \text{ g/cm}^3$

$= 0.06243 \text{ lb}_m/\text{ft}^3$

$10^3 \text{ kg/m}^3 = 1 \text{ g/cm}^3 = 62.428 \text{ lb}_m/\text{ft}^3$

Volumetric Flow

$1 \text{ m}^3/\text{s} = 35.3145 \text{ ft}^3/\text{s} = 15{,}850.2 \text{ gal/min (gpm)}$

$1 \text{ gpm} = 6.30907 \times 10^{-5} \text{ m}^3/\text{s} = 2.22802 \times 10^{-3} \text{ ft}^3/\text{s} = 3.7854 \text{ liter/min}$

$1 \text{ liter/min} = 0.26417 \text{ gpm}$

Ver. 21-Sep-2011

| Temperature | $T(^oC) = \frac{5}{9}\left[T(^oF) - 32\right]$ |
| | $T(^oF) = \frac{9}{5}T(^oC) + 32 = 1.8T(^oC) + 32$ |

Absolute Temperature
$T(K) = T(^\circ C) + 273.15$
$T(^\circ R) = T(^\circ F) + 459.67$

Temperature Interval (ΔT)
$1\ C^\circ = 1\ K = 1.8\ F^\circ = 1.8\ R^\circ$
$1 F^\circ = 1\ R^\circ = (5/9)\ C^\circ = (5/9)\ K$

USEFUL QUANTITIES

$SG = \rho(20°C)/\rho_{water}(4°C)$

$\rho_{water}(4°C) = 1000\ kg/m^3 = 62.43\ lb_m/ft^3 = 1.000\ g/cm^3$
$\rho_{water}(25°C) = 997.08\ kg/m^3 = 62.25\ lb_m/ft^3 = 0.99709\ g/cm^3$

$g = 9.8066\ m/s^2 = 980.66\ cm/s^2 = 32.174\ ft/s^2$

$\mu_{water}(25°C) = 8.937 \times 10^{-4}\ Pa \cdot s = 8.937 \times 10^{-4}\ kg/m \cdot s$
$= 0.8937\ cp = 0.8937 \times 10^{-2}\ g/cm \cdot s = 6.005 \times 10^{-4}\ lb_m/ft \cdot s$

Composition of air:

N_2	78.03%	
O_2	20.99%	
Ar	0.94%	
CO_2	0.03%	
H_2, He, Ne, Kr, Xe	0.01%	
	100.00%	

$M_{air} = 29\ g/mol = 29\ kg/kmol = 29\ lb_m/lbmole$

$\hat{C}_{p,water}(25°C) = 4.182\ kJ/kg\ K = 0.9989\ cal/g°C = 0.9997\ Btu/lb_m°F$

$R = 8.314\ m^3 \cdot Pa/mol \cdot K = 0.08314\ liter \cdot bar/mol \cdot K = 0.08206\ liter \cdot atm/mol \cdot K$
$= 62.36\ liter \cdot mm\ Hg/mol \cdot K = 0.7302\ ft^3 \cdot atm/lbmole \cdot °R$
$= 10.73\ ft^3 \cdot psia/lbmole \cdot °R$
$= 8.314\ J/mol \cdot K$
$= 1.987\ cal/mol \cdot K = 1.987\ Btu/lbmole \cdot °R$

Data Correlations for Examinations

CM3110 Transport Phenomena I
Michigan Technological University
Professor Faith A. Morrison

I.   Flow through Smooth Pipes

A.   All Reynolds numbers: Morrison

The correlation from Morrison (2013) fits the smooth pipe data for all Reynolds numbers; beyond $Re = 4000$ this correlation follows the Prandtl equation (see Figure 1; Morrison, equation 7.158). This correlation is explicit in f; when flow rate is known, Δp may be found directly; when Δp is known, Q or $\langle v \rangle$ must be solved for iteratively.

Morrison (2013)
$$f = \left(\frac{0.0076 \left(\frac{3170}{Re} \right)^{0.165}}{1 + \left(\frac{3170}{Re} \right)^{7.0}} \right) + \frac{16}{Re} \qquad (1)$$

B. $4,000 \leq Re \leq 1 \times 10^6$: Prandtl

The Prandtl correlation for $f(Re)$ in turbulent flow is not explicit in friction factor and must be solved iteratively except when f is known (Morrison, equation 7.156). This is good only for $Re > 4,000/$

Prandtl
or VonKarman-Nikuradse
(Denn, 1980)
$$\frac{1}{\sqrt{f}} = 4.0 \log \left(Re \sqrt{f} \right) - 0.40 \qquad (2)$$

C. $4,000 \leq Re \leq 1 \times 10^6$: A simplified Correlation

For the turbulent regime, an approximate correlation that is much simpler to work with (with a calculator on an exam, for example) is given here and shown in Figure 2 (Morrison, equation 7.157). This is good only for $Re > 4,000$.

Simplified Turbulent
(White, 1974)
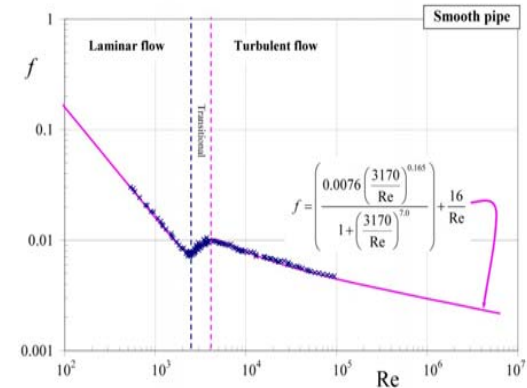$$f = \frac{1.02}{4} (\log Re)^{-2.5} \qquad (3)$$

Figure 1: Equation 3 captures smooth pipe friction factor as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are Nikuradse's experimental data for flow in smooth pipes (Nikuradse, 1933). Use beyond $Re = 10^6$ is not recommended; for Re>4000 equation 3 follows the Prandtl equation.
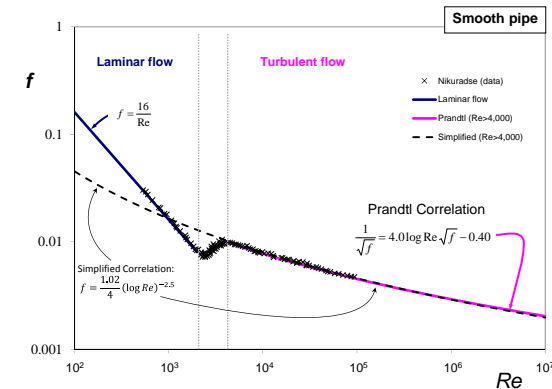

Figure 2: For turbulent flow, the simplified (equation 3) or Prandtl (equation 2) correlations may be used. For work with a calculator, the simplified correlation is perhaps the easiest to work with.

II.   Flow Around a Sphere

A.   All Reynolds Numbers: Morrison

The correlation from Morrison (2013) fits the flow around a sphere for all Reynolds numbers (Figure 3; Morrison equation 8.83); beyond $Re = 10^6$ this correlation follows the curve shown in Figure 3.

$$\text{Morrison (2013)} \qquad C_D = \frac{24}{Re} + \frac{2.6\left(\frac{Re}{5.0}\right)}{1 + \left(\frac{Re}{5.0}\right)^{1.52}} + \frac{0.411\left(\frac{Re}{263,000}\right)^{-7.94}}{1 + \left(\frac{Re}{236,000}\right)^{-8.00}} + \frac{0.25\left(\frac{Re}{10^6}\right)}{1 + \left(\frac{Re}{10^6}\right)} \qquad (4)$$

Simplified Correlations

The correlations below (Morrison, 2013; equation 8.82) are simpler relationships more suitable to calculator/exam work.

$$Re < 2 \qquad\qquad C_D = \frac{24}{Re} \qquad\qquad (5)$$

$$0.1 \leq Re \leq 1,000 \qquad C_D = \frac{24}{Re}(1 + 0.14Re^{0.7}) \qquad (6)$$

$$1,000 \leq Re \leq 2.6 \times 10^5 \qquad C_D = 0.445 \qquad (7)$$

$$2.8 \times 10^5 \leq Re \leq 10^6 \qquad \frac{\log C_D}{\left(\frac{Re}{10^6}\right)} = 4.43 \log Re - 27.3 \qquad (8)$$
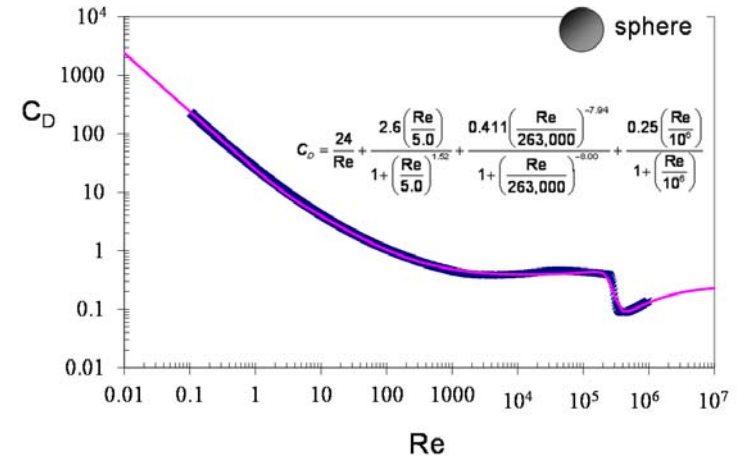

Figure 3: Equation 4 captures flow around a sphere as a function of Reynolds number over the entire Reynolds-number range (Morrison, 2013) and is recommended for spreadsheet use. Also shown are experimental data from White (1974). Use beyond $Re = 10^6$ is not recommended.

References
M. Denn, Process Fluid Mechanics (Prentice Hall, Englewood Cliffs, NJ, 1980)
F. A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press, New York, 2013).
F. M. White, *Viscous Fluid Flow* (McGraw-Hill, Inc.: New York, 1974).

The Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

CM3110 Spring 2002 Faith A. Morrison

Cartesian coordinates

$$
\begin{pmatrix}
\tau_{xx} & \tau_{xy} & \tau_{xz} \\
\tau_{yx} & \tau_{yy} & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & \tau_{zz}
\end{pmatrix}_{xyz}
= \mu
\begin{pmatrix}
2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\
\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \\
\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z}
\end{pmatrix}_{xyz}
\qquad \text{1-1}
$$

Cylindrical coordinates

$$
\begin{pmatrix}
\tau_{rr} & \tau_{r\theta} & \tau_{rz} \\
\tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\
\tau_{zr} & \tau_{z\theta} & \tau_{zz}
\end{pmatrix}_{r\theta z}
= \mu
\begin{pmatrix}
2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\
r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \\
\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z}
\end{pmatrix}_{r\theta z}
\qquad \text{1-2}
$$

Spherical coordinates

$$
\begin{pmatrix}
\tau_{rr} & \tau_{r\theta} & \tau_{r\phi} \\
\tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta\phi} \\
\tau_{\phi r} & \tau_{\phi\theta} & \tau_{\phi\phi}
\end{pmatrix}_{r\theta\phi}
=
$$

$$
\mu
\begin{pmatrix}
2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\
r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\
\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta\cot\theta}{r}\right)
\end{pmatrix}_{r\theta\phi}
\qquad \text{1-3}
$$

These expressions are general and are applicable to three-dimensional flows. For unidirectional flows they reduce to Newton's law of Viscosity. Reference: Faith A. Morrison, *Understanding Rheology* (Oxford University Press: New York, 2001).

Table 1.4. Published friction-loss factors for turbulent flow through valves, fittings, expansions, and contractions

Fitting	Friction-loss factor, K_f
Standard elbow, 45°	0.35
Standard elbow, 90°	0.75
Tee used as ell	1.0
Tee, branch blanked off	0.4
Return bend	1.5
Coupling	0.04
Union	0.04
Gate valve, wide open	0.17
Gate valve, half open	4.5
Globe valve, bevel seat, wide open	6.0
Globe valve, bevel seat, half open	9.5
Check valve, ball	70.0
Check valve, swing	2.0
Water meter, disk	7.0
Expansion from A_1 to A_2	$\left(1 - \dfrac{A_1}{A_2}\right)^2$
Contraction from A_1 to A_2	$0.55\left(1 - \dfrac{A_2}{A_1}\right)$

Source: Perry's Handbook [132]

Table 1.5. Friction-loss factors K_f for laminar flow through selected valves, fittings, expansions and contractions

| Fitting | K_f | | | | | |
	$Re_i = 50$	100	200	400	1,000	Turbulent
Elbow, 90°	17	7	2.5	1.2	0.85	0.75
Tee	9	4.8	3.0	2.0	1.4	1.0
Globe valve	28	22	17	14	10	6.0
Check valve, swing	55	17	9	5.8	3.2	2.0
Expansion from A_1 to A_2			$2\left(1 - \dfrac{A_1}{A_2}\right)^2$			$\left(1 - \dfrac{A_1}{A_2}\right)^2$
Contraction from A_1 to A_2			$\dfrac{0.55}{0.5}\left(1 - \dfrac{A_2}{A_1}\right)$			$0.55\left(1 - \dfrac{A_2}{A_1}\right)$

Source: Perry's Handbook [132]