First problem in Fluid Mechanics: Flow down an inclined plane.

When we're asked for the velocity field or velocity distribution, we must perform a microscopic balance.
EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

What is the velocity field in the steady flow of water down a slope that is wide and long. The fluid properties are constant, and the flow is driven by gravity. The flow is slow so that no waves are formed. What is the force on the surface due to the water flow? What is the flow rate?
EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

- fully developed flow
- steady state
- flow in layers (laminar)

First - Choose a coordinate system
\( \text{mass}_g = \text{Force due to Gravity} \)

\( U_2 \) (position)
What is $g$ in our coordinate system?

\[ g = \begin{pmatrix} g \sin \beta \\ 0 \\ g \cos \beta \end{pmatrix} \times 2 \]

\[ g_x \hat{e}_x + g_z \hat{e}_z = g \]

\[ g_x = 3.5 \sin \beta \quad g_z = g \cos \beta \]
Mass balance, flowing system
(open system; control volume):

\[
\begin{align*}
\{ \text{net mass} \\ \text{flowing in} \} = \{ \text{rate of} \\
\text{accumulation} \\ \text{of mass} \} \\
\Sigma_{in} - \Sigma_{out} = \text{steady state}
\end{align*}
\]
\( (x + \Delta x, y + \Delta y, z + \Delta z) \)

\( v_z(x) \)

\( \beta \)

inlet

\( (x, y, z) \)

Outlet

Surface
**Mass Bal**

mass flow in = mass flow out

\[
\text{mass rate} \times \text{time} = \int \rho \, d\tau \cdot \int dA \, d\tau
\]

\[
\int \frac{\partial \rho}{\partial \tau} \, d\tau = \rho \int \frac{\partial \tau}{\partial \tau} \, d\tau
\]

\[
\frac{\partial \rho}{\partial \tau} \cdot \frac{\partial \tau}{\partial \tau} = \rho
\]

\[
\frac{\partial \tau}{\partial \tau} = 1
\]

\[
\frac{\partial \rho}{\partial \tau} = \rho
\]

\[
\frac{\partial \tau}{\partial \tau} = \frac{\partial \rho}{\partial \tau}
\]
Momentum balance, flowing system
(open system; control volume):

\[ \sum f = ma \implies \]

\[ \sum \text{all forces} \]

\[ \begin{align*}
\text{sum of forces} & \quad \text{acting on control vol} \\
\text{net momentum} & \quad \text{flowing in} \\
\sum_{\text{in}} & \quad - \sum_{\text{out}} \\
\end{align*} \]

\[ = \left\{ \begin{array}{c}
\text{rate of} \\
\text{accumulation} \\
\text{of momentum} \\
\text{steady state}
\end{array} \right\} \]

\[ \sum F_{on_i} + \sum \left\{ \begin{array}{c}
momentum \\
\text{flowing in} \\
in the streams \\
\end{array} \right\} - \sum \left\{ \begin{array}{c}
momentum \\
\text{flowing out} \\
in the streams \\
\end{array} \right\} = 0 \]
Momentum Balance

\[ \sum F = \text{momentum in} - \text{momentum out} = 0 \]

\[
\text{momentum in} = (\text{mass})(\text{velocity}) \times \left( \frac{\text{momentum}}{\text{volume}} \right) \times \frac{\text{time}}{\text{time}}
\]

\[
= \left( P \left( \frac{v_z}{z} \right) \right) (\Delta x \Delta y)
\]

\[
= P \left( \frac{v_z}{z} \right)^2 \Delta x \Delta y
\]
momentum = \( (p_{out} \times \text{velocity}) \times \text{mass} \)

\[ \sum F_{on} = \text{(gravity)} + \text{(viscous forces)} \]

\( \sum F_{on} \) sum of forces "m" c.v.

Force due to gravity = \( g \cdot \Delta x \Delta y \Delta t \)

acceleration \cdot \text{mass} \cdot \text{force}
Force due to gravity on C.V. is given by:

\[ F_{gz} = \int \int \int_{V} \rho g \, \text{d}x \, \text{d}y \, \text{d}z \]

\[ = \int \int \int_{V} \rho g \sin \beta \, \text{d}x \, \text{d}y \, \text{d}z \]

\[ = \int \int \int_{V} \rho g \cos \beta \, \text{d}x \, \text{d}y \, \text{d}z \]

\[ F_{gz} = \int \int \int_{V} \rho g \cos \beta \, \text{d}x \, \text{d}y \, \text{d}z \]

\[ \text{(Gravity force on C.V.)} \]

\[ \text{(Component in z-direction)} \]
MORE FORCES:

**Viscous Forces**

- Top:
  \[
  \left( -\tau \right)_{x^2} \Delta z \Delta y
  \]

- Bottom:
  \[
  \left( \tau \right)_{x^2} \Delta z \Delta y
  \]

"force on an 'x' surface in the 'z' direction" 

"needed for our sign convention - goes with 'tilde t' or, 'flux of 'z' momentum in 'x' direction"
CALC I

The definition of derivative

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{df}{dx}
\]

or,

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{df}{dx}
\]

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Assemble the Micro Momentum Eq.

\[ 0 = F_{\text{gravity}} + F_{\text{viscous}} + \text{momentum convected in} - \text{momentum convected out} \]

\[ 0 = \Delta x \Delta y \Delta z \rho \cos \beta + \left( -\frac{\partial p}{\partial z} \right) \Delta z \Delta y \]

\[ - \left( -\frac{\partial p}{\partial x} \right) \Delta x \Delta y + \rho \left( \frac{\partial v_z}{\partial z} \right)^2 \Delta x \Delta y \]

\[ - \rho \left( \frac{\partial v_z}{\partial z} + \frac{\partial v_x}{\partial z} \right)^2 \Delta x \Delta y \]
Divide by $\Delta x \Delta z$:

$$0 = \rho g \cos \beta + \left( -\frac{\frac{\Delta z}{\Delta x}}{x} \right) \frac{\Delta x}{\Delta x} - \left( -\frac{\frac{\Delta z}{\Delta x}}{x+\Delta x} \right) \frac{\Delta x}{\Delta x}$$

$$+ \rho \left( \frac{\frac{\Delta z}{\Delta x}}{\Delta z} \right)^2 - \rho \left( \frac{\frac{\Delta z}{\Delta x}}{\Delta z+\Delta z} \right)^2$$

$$= 0 \quad \text{since} \quad \frac{\frac{\Delta z}{\Delta x}}{\Delta z} = \frac{\Delta z}{\Delta z} \quad \text{at} \quad \Delta z = \Delta z$$
\[
\lim_{\Delta x \to 0} \frac{c_{x_2}^2 (x + \Delta x) - c_{x_2}^2}{\Delta x} = -p g \cos \beta
\]

\[
\frac{d}{dx} (c_{x_2}^2) = -p g \cos \beta
\]

\[
c_{x_2} = (-p g \cos \beta) x + c_1
\]
Newton's Law of Viscosity

Momentum in \( \tau \)-dir
Momentum flux in \( x \)-dir.

\[
\nabla \tau_{xz} = \mu \frac{dV_z}{dx}
\]

\[
\nabla \tau_{xz} = \mu \frac{dV_z}{dx} = \left(-\rho g \cos \beta \right) x + c_1
\]

\[
\frac{dV_z}{dx} = \left(-\rho g \cos \beta \right) x + \frac{c_1}{\mu}
\]
\( v_e = \left( -\frac{p g \cos \beta}{\mu} \right) \frac{x^2}{2} + \left( \frac{c_1}{\mu} \right) x + c_2 \)

**Boundary Conditions:**

- No slip at the wall
- \( x = H \) \( (H = \text{film thickness}) \)
- \( v_e = 0 \)

\( 0 = -\frac{p g \cos \beta}{2\mu} \frac{H^2}{2} + \frac{c_1 H}{\mu} + c_2 \)

One eqn, 2 unknowns
2nd B.C. →

\[ x = 0 \]

\[ \left. \frac{dT}{dx} \right|_{\text{air}} = \left. \frac{dT}{dx} \right|_{\text{water}} \]

\[ \left. \frac{dV}{dx} \right|_{\text{air}} \]

\[ \left. \frac{dV}{dx} \right|_{\text{water}} \]

\[ \frac{dV}{dx} \bigg|_{x=0} = 0 \]

\[ \text{Air is very small} \]
\[ c_1 = 0 \]

\[ \Rightarrow \quad 0 = -\frac{pg \cos \beta H^2}{2h} + c_2 \]

\[ c_2 = \frac{pg \cos \beta H^2}{2h} \]
2nd BC:

\[ x = 0 \]

\[ \frac{dV_z}{dx} = 0 \]

\[ \phi \frac{g \cos \beta}{\sqrt{\lambda}} \cdot x + \frac{c_1}{r} \]

\[ c_1 = 0 \]
To finish the problem, we submit the solved for values of $C_1$ and $C_2$ into our $V_z(r)$ equation, and simplify.

Page 19:

$$V_z = \left( -\frac{\rho g \cos \beta}{2 \mu} \right) x^2 + \left( \frac{C_1}{\mu} \right) x + C_2$$

$c_1 = 0$ \(\text{(pg 21)}\)

$$c_2 = \frac{\rho g \cos \beta H^2}{2 \mu}$$

$$V_z = \frac{\rho g \cos \beta}{2 \mu} \left( -x^2 + H^2 \right)$$

$$V_z(r) = \frac{\rho g \cos \beta}{2 \mu} \left( H^2 - x^2 \right)$$

(matches lecture 445, slide 32)
What is the flow rate?

\[ Q = \int \int \int \mathbf{u} \cdot d\mathbf{x} \, dy \]

gives the flow rate

\[ \frac{Q}{HW} = \langle \mathbf{u} \rangle \]

gives the cross-sectional area

\[ \int \int d\mathbf{x} \, dy = HW \]
What is the force on the plane surface? (yz-plane)

unit normal \( \hat{n} = -\hat{e}_x \)

\[ L = \text{length} \]

\[ \hat{e}_x = \hat{n} \]

\[ dy \]

\[ dz \]

\[ \text{Force} = \left( \frac{\text{FORCE}}{a \pi c} \right) (a \pi c) \]

\[ F_z = \int_0^L \int_0^W \tilde{c} x^2 \, dy \, dx \]

wetted Surface

\[ W = \text{width} \]
From p/7 (W/C_i = 0):

\[ F_x = -p g \cos \beta x \]

at \( x = H \)

\[ F_x(A) = -p g \cos \beta H \]

**Constant**

\[ F_x = \int_0^L \int_0^H t_{x2} \bigg|_{x=H} \, dy \, dz \]

\[ = \left( -p g \cos \beta H \right) \int_0^L \int_0^H \, dy \, dz \]

*Force on fluid:*

\[ F_x = -p g \cos \beta H L W \]

Force on plane = \(-F_x\)