Macroscopic Balances

- **Mass**
- **Momentum**
- **Energy (mechanical energy)**

Plan of Attack:

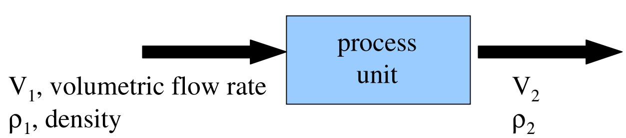
- Describe
- Derive (mass, momentum)
- Apply to example problems

Calculate forces on walls, size pumps, calculate fluid friction, pressure drops, etc.

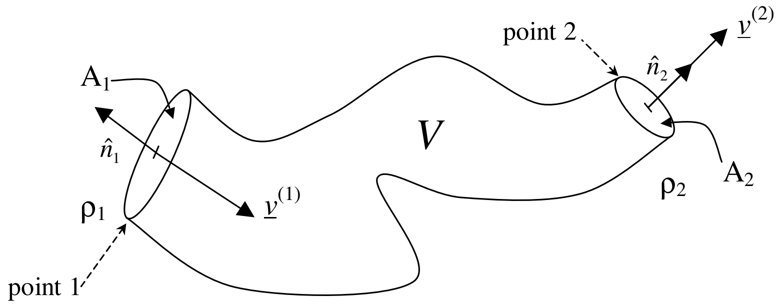
Macroscopic Balances

- Use when we do not need the details of the velocity profile
- 3 types:
  - mass
  - momentum
  - energy

Macroscopic Mass Balance:



Arbitrary, single-input, single-output system: *special case* of velocity perpendicular to areas



Assumptions:
- steady state
- single-input, single output
- **$\underline{v}^{(i)}$ perpendicular to A_i**
- ρ constant across surface

Macroscopic Mass Balance:

$$\boxed{\begin{gathered} mass\ in = mass\ out \\ \rho_1 \left\langle v^{(1)} \right\rangle A_1 = \rho_2 \left\langle v^{(2)} \right\rangle A_2 \end{gathered}}$$

average velocity through surface A_1

cross-sectional area, in

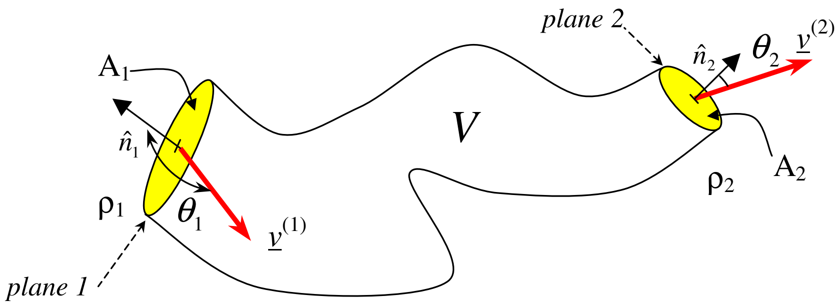
average velocity through surface A_2

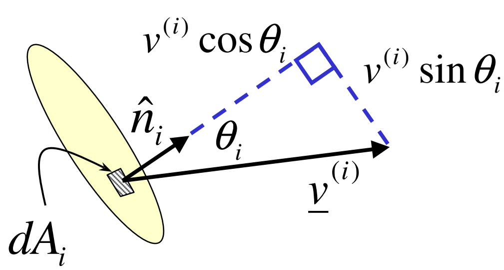
cross-sectional area, out

Assumptions:
- steady state
- single-input, single output
- **$\underline{v}^{(i)}$ perpendicular to A_i**
- ρ constant across surface

Arbitrary, single-input, single-output system:

velocity is *NOT perpendicular* to cross-sectional areas



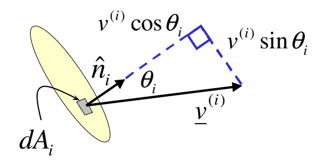


$\hat{n}_i$ = ***outwardly* pointing unit normal**

Macroscopic Mass Balance:

$$\boxed{\begin{gathered} 0 = net\ mass\ out \\ 0 = \rho_1 \left\langle v^{(1)} \right\rangle \cos\theta_1 A_1 + \rho_2 \left\langle v^{(2)} \right\rangle \cos\theta_2 A_2 \end{gathered}}$$

Assumptions:
- steady state
- single-input, single output
- **$\underline{v}^{(i)}$ NOT perpendicular to A_i**
- ρ_i constant across surface



$\hat{n}_i$ = ***outwardly*** **pointing unit normal**

Macroscopic Momentum Balance:

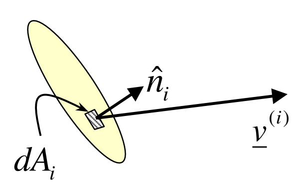
$$\begin{pmatrix} net\ momentum \\ convected\ in \end{pmatrix} + \sum forces_{on\ fluid} = \cancel{\begin{pmatrix} rate\ of \\ accumulation \\ of\ momentum \end{pmatrix}}$$

steady state

$$\begin{pmatrix} net\ momentum \\ convected\ in \end{pmatrix} = \sum_{\substack{inlets,\\ outlets}} \left(\frac{momentum}{volume} \right) \left(\frac{volume\ flow\ \underline{in}}{time} \right)$$

Since the velocity varies across the surfaces A_1 and A_2, we need to consider the flow across a small area, dA_i, which we will then integrate over each A_i.

The velocity at position dA is $\underline{v}^{(i)}$

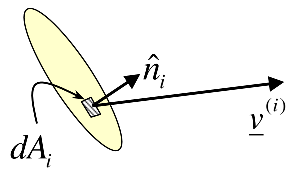


Macroscopic Momentum Balance
(continued)

$$\begin{pmatrix} net\ momentum \\ convected\ in \end{pmatrix} + \sum forces_{on\ fluid} = 0$$

$$\begin{pmatrix} net\ momentum \\ convected\ in \end{pmatrix} = \sum_{\substack{inlets, \\ outlets}} \left(\frac{momentum}{volume} \right) \left(\frac{volume\ \ flow\ \underline{in}}{time} \right)$$

$$\boxed{\rho_i \underline{v}^{(i)}} \qquad \boxed{-\hat{n}_i \cdot \underline{v}^{(i)}\ dA_i}$$



$$= \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)} \right) \left(-\hat{n}_i \cdot \underline{v}^{(i)}\ dA_i \right)$$

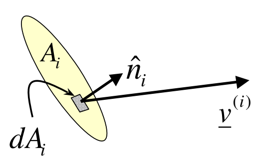
Macroscopic Momentum Balance
(continued)

$$\underline{v}^{(i)} = v^{(i)} \hat{v}^{(i)}$$

$$\begin{pmatrix} net\ momentum \\ convected\ in \end{pmatrix} = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)} \right) \left(- \underbrace{\hat{n}_i \cdot \underline{v}^{(i)}}_{v^{(i)} \cos\theta_i}\ dA_i \right)$$

$$= \sum_i \iint_{A_i} \left(\rho_i v^{(i)} \hat{v}^{(i)} \right) \left(-v^{(i)} \cos\theta_i\ dA_i \right)$$

$$= \sum_i -\rho_i \hat{v}^{(i)} \cos\theta_i \left(\iint_{A_i} \left(v^{(i)} \right)^2 dA_i \right)$$



We have assumed that the direction of $\underline{v}^{(i)}$ does not vary across A_i.

Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **NOT perpendicular to A_i**
- ρ_i constant across surfaces
- $\hat{v}^{(i)}$ constant across surfaces

Macroscopic Momentum Balance, *continued*

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \left[\iint_{A_1} \left(v^{(1)}\right)^2 dA_1 \right] - \rho_2 \cos\theta_2 \hat{v}^{(2)} \left[\iint_{A_2} \left(v^{(2)}\right)^2 dA_2 \right] + \sum_i \underline{F}_{i,on}$$

Recall that the average of a function f is calculated from:

$$\langle f(x,y) \rangle = \frac{\iint_A f \, dA}{\iint_A dA} = \frac{1}{A} \iint_A f \, dA$$

Macroscopic Momentum Balance, *continued*

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \underbrace{\left[\iint_{A_1} \left(v^{(1)}\right)^2 dA_1 \right]}_{= \left\langle \left(v^{(1)}\right)^2 \right\rangle A_1} - \rho_2 \cos\theta_2 \hat{v}^{(2)} \underbrace{\left[\iint_{A_2} \left(v^{(2)}\right)^2 dA_2 \right]}_{= \left\langle \left(v^{(2)}\right)^2 \right\rangle A_2} + \sum_i \underline{F}_{i,on}$$

$$\boxed{0 = -\rho_1 A_1 \left\langle \left(v^{(1)}\right)^2 \right\rangle \cos\theta_1 \hat{v}^{(1)} - \rho_2 A_2 \left\langle \left(v^{(2)}\right)^2 \right\rangle \cos\theta_2 \hat{v}^{(2)} + \sum_i \underline{F}_{i,on}}$$

But what is this?

We can make this look more like other convective terms we have seen by introducing a factor relating $\langle v^2 \rangle$ to average velocity squared.

β quantifies the variation of the true velocity profile from **plug flow** (flat profile). β = ***Velocity Correction Factor***

$$0 = -\rho_1 A_1 \left\langle \left(v^{(1)}\right)^2 \right\rangle \cos\theta_1 \hat{v}^{(1)} - \rho_2 A_2 \left\langle \left(v^{(2)}\right)^2 \right\rangle \cos\theta_2 \hat{v}^{(2)} + \sum_i \underline{F}_{i,on}$$

define: $\boxed{\beta \equiv \frac{\langle v \rangle^2}{\langle v^2 \rangle}}$

experimental result

$\beta_{turbulent} = 0.95\text{-}0.99$

$\beta_{laminar} = 0.75$

Final Result: Macroscopic Momentum Balance

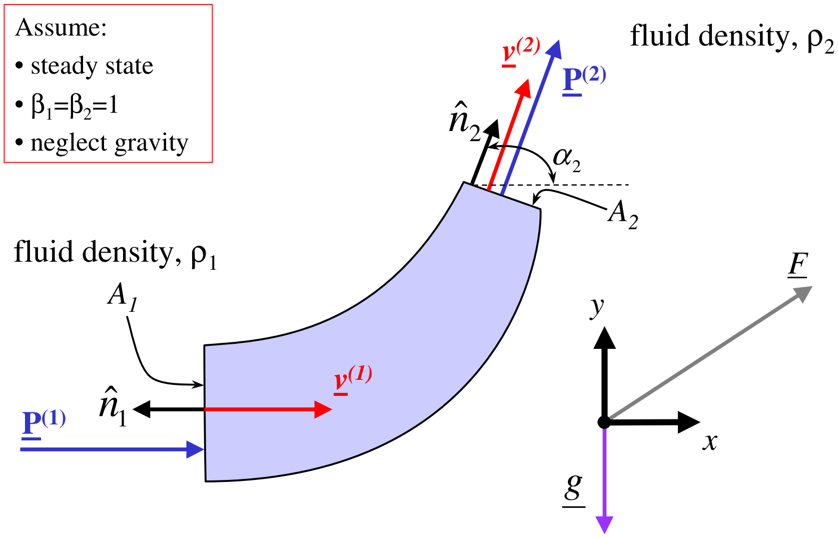
$$\boxed{0 = \frac{-\rho_1 A_1 \langle v^{(1)} \rangle^2 \cos\theta_1}{\beta_1} \hat{v}^{(1)} + \frac{-\rho_2 A_2 \langle v^{(2)} \rangle^2 \cos\theta_2}{\beta_2} \hat{v}^{(2)} + \sum_i \underline{F}_{i,on}}$$

vector equation

Macroscopic Momentum Balance Example:

Calculate the force on a reducing bend

Assume:
- steady state
- $\beta_1 = \beta_2 = 1$
- neglect gravity



Types of Momentum Transfer

Microscopic	**Macroscopic**
convection *(momentum flows in)*	convection *(momentum flows in)*
pressure forces	pressure forces
viscous forces *(or viscous flux)*	wall forces *(due to viscosity)*
body forces *(gravity)*	body forces *(gravity)*

After calculating the flow field with microscopic balances you can calculate wall forces

With macroscopic balances you can often **calculate wall forces directly**

Problem-Solving Procedure - Macroscopic Momentum Problems

$$0 = -\left[\frac{m_1 \left\langle v^{(1)} \right\rangle \cos\theta_1}{\beta_1}\right]\hat{v}^{(1)} - \left[\frac{m_2 \left\langle v^{(2)} \right\rangle \cos\theta_2}{\beta_2}\right]\hat{v}^{(2)} + \sum_i \underline{F}_{i,on}$$

1. sketch system; choose system on which you will perform balance
2. choose coordinate system
3. perform macroscopic mass balance

Consider angles carefully

4. perform macroscopic momentum balance (vector equation; forces are pressure, gravity, force on the wall; all forces ON the system)
5. solve (usually for force on the wall)