Unsteady State Heat Transfer

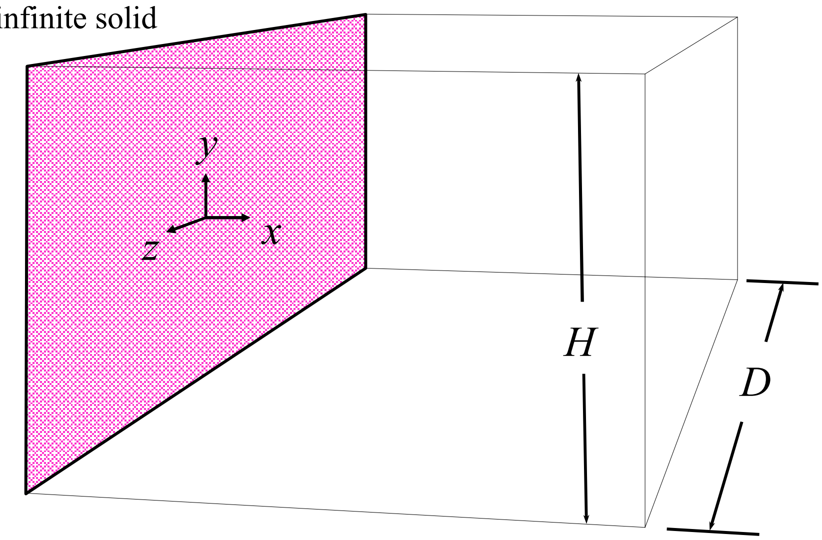
Use same microscopic energy balance eqn as before.

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v} \cdot \nabla T$: convection
- $k \nabla^2 T$: conduction (all directions)
- S: source (energy generated per unit volume per time)

see handout for component notation

Example 1: Unsteady Heat Conduction in a Semi-infinite solid



H, D, very large

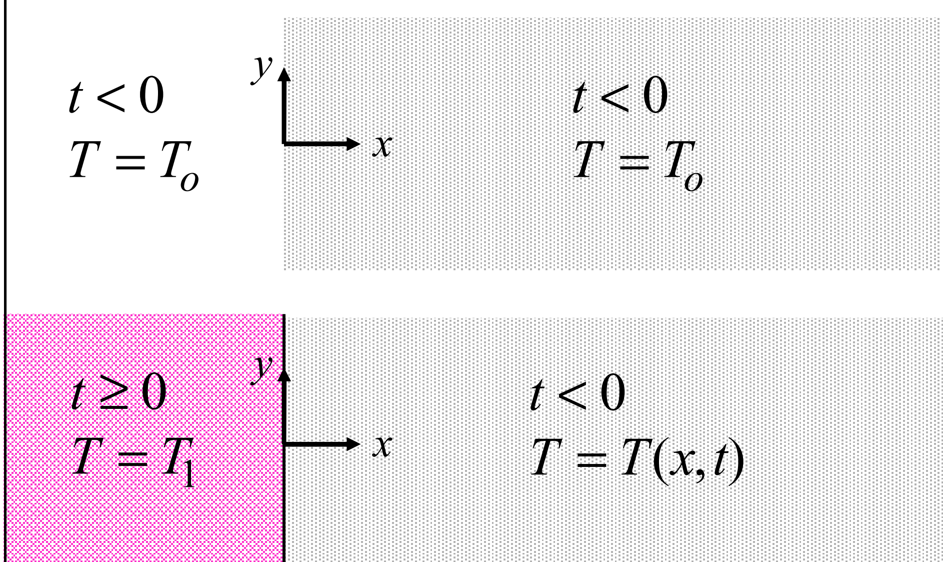
Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature T_o. At time $t = 0$, the left face of the slab is exposed to an environment at temperature T_1. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, k, density, ρ, and heat capacity, C_p.

Newton's law of cooling BC's:

$$q_x = hA(T_{bulk} - T_{surface})$$

Initial Condition:



Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} = \text{ thermal diffusivity}$$

what are the boundary conditions? initial conditions?

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p} \left(\frac{\partial^2 T}{\partial x^2} \right) = \alpha \left(\frac{\partial^2 T}{\partial x^2} \right)$$

Initial condition: $t = 0, T = T_o \; \forall \; x$

Boundary conditions:

$$x = 0, \quad q_x = hA(T - T_1) \quad \forall \; t > 0$$

$$x = \infty, \quad T = T_o \quad \forall \; t$$

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T-T_o}{T_1-T_o}\right) = erfc(\zeta) - e^{\beta(2\zeta+\beta)} erfc(\zeta+\beta)$$

Geankoplis 4th ed., eqn 5.3-7, page 363

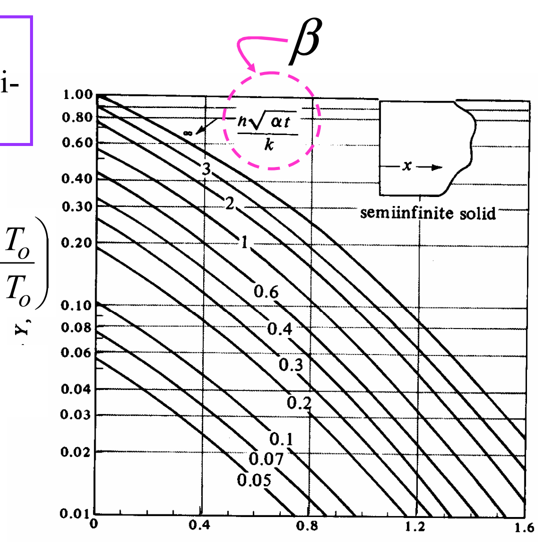
$$\beta \equiv \frac{h\sqrt{\alpha\, t}}{k} \qquad \zeta \equiv \frac{x}{2\sqrt{\alpha\, t}}$$

complementary error function $\quad erfc(x) = 1 - erf(x)$

error function $\quad erf(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-(x')^2} dx'$

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\left(\frac{T-T_o}{T_1-T_o}\right)$$



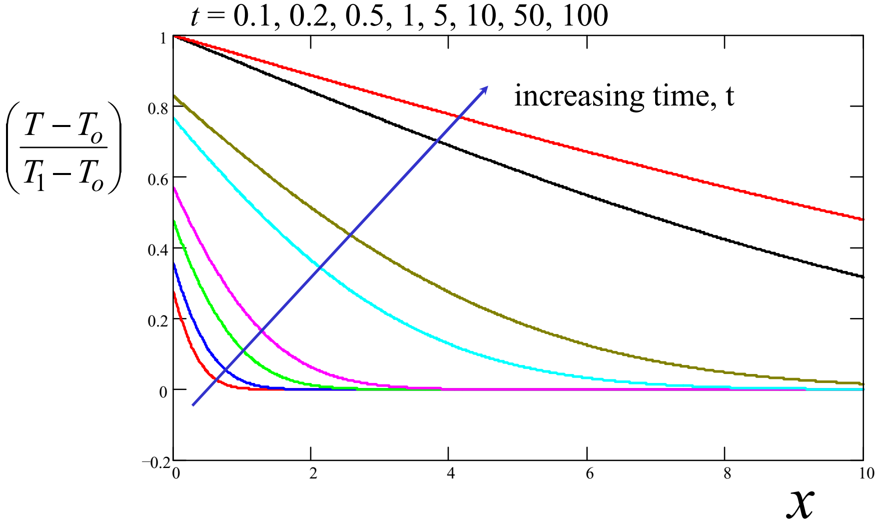
$$\zeta = \frac{x}{2\sqrt{\alpha\, t}}$$

Geankoplis 4th ed., Figure 5.3-3, page 364

Unsteady State Heat Conduction in a Semi-Infinite Slab

$h = k = \alpha = 1$

$t = 0.1, 0.2, 0.5, 1, 5, 10, 50, 100$



evaluated on Mathcad Plus

How could we use this solution?

Example 2: Will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground? Use the following physical properties:

$$h = 2.0 \frac{BTU}{h\ ft^2\ {}^oF}$$

$$\alpha_{soil} = 0.018 \frac{ft^2}{h}$$

$$k_{soil} = 0.5 \frac{BTU}{h\ ft\ {}^oF}$$