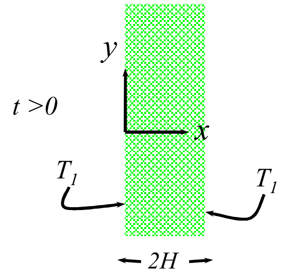
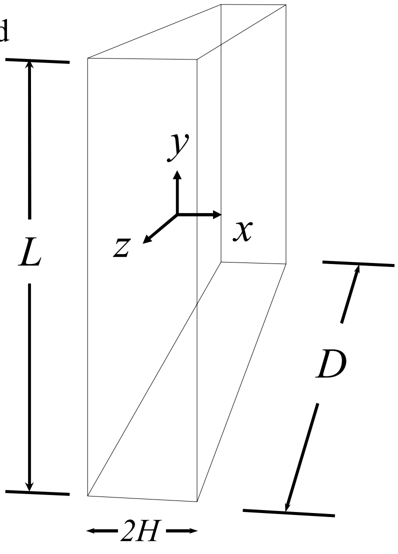
Example 1: Unsteady Heat Conduction in a Finite-sized solid

- The slab is tall and wide, but of thickness $2H$
- Initially at T_o
- at time $t = 0$ the temperature of the sides is changed to T_1





Unsteady State Heat Transfer

Use same microscopic energy balance eqn as before.

$$\rho \hat{C}_p \left(\underbrace{\frac{\partial T}{\partial t}}_{\text{rate of change}} + \overbrace{\underline{v} \cdot \nabla T}^{\text{convection}} \right) = \underbrace{k \nabla^2 T}_{\substack{\text{conduction}\\ \text{(all directions)}}} + \overbrace{S}^{\text{source}}$$

(energy generated per unit volume per time)

see handout for component notation

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t}+v_x\frac{\partial T}{\partial x}+v_y\frac{\partial T}{\partial y}+v_z\frac{\partial T}{\partial z}=\frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\frac{S}{\rho\hat{C}_p}$$

$$\alpha\equiv\frac{k}{\rho\hat{C}_p}=\text{ thermal diffusivity}$$

what are the boundary conditions? initial conditions?

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Unsteady State Heat Conduction in a Finite Slab

$$\frac{\partial T}{\partial t}=\frac{k}{\rho\hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2}\right)=\alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

Initial condition: $t=0, T=T_o \; \forall \; x$

Boundary conditions:

$$\left.\begin{array}{ll} x=0, & T=T_1 \\ x=2H, & T=T_1 \end{array}\right\} \quad \forall \; t>0$$

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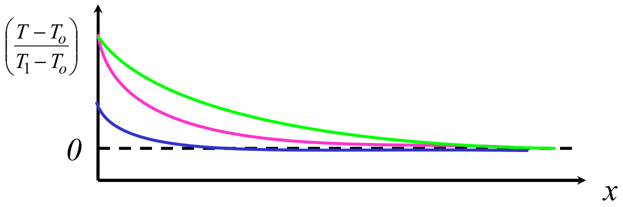
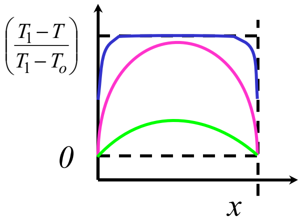
Q: *How can two completely different situations give the same governing equation?*

A: The boundary conditions make all the difference





For more solutions to this equation see Carslaw and Jeager, Conduction of Heat in Solids, 2nd Edition, Oxford, 1959

Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

Let $Y \equiv \left(\frac{T_1 - T}{T_1 - T_o}\right)$

$$\boxed{\frac{\partial Y}{\partial t} = \alpha\left(\frac{\partial^2 Y}{\partial x^2}\right)}$$

Guess: $Y = X(x)\Theta(t)$

Initial condition:

$$t = 0, T = T_o\ \forall\ x \Rightarrow Y = 1$$

Boundary conditions:

$$\left.\begin{array}{ll} x = 0, & T = T_1 \Rightarrow Y = 0 \\ x = 2H, & T = T_1 \Rightarrow Y = 0 \end{array}\right\}\ \forall\ t > 0$$

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$Y = X(x)\Theta(t) \qquad \frac{\partial Y}{\partial t} = \alpha\left(\frac{\partial^2 Y}{\partial x^2}\right)$$

$$\frac{\partial Y}{\partial t} = \frac{\partial}{\partial t}\big(X(x)\Theta(t)\big) = X(x)\frac{d\Theta(t)}{dt}$$

$$\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x}\big(X(x)\Theta(t)\big) = \frac{dX(x)}{dx}\Theta(t)$$

$$\frac{\partial^2 Y}{\partial x^2} = \frac{d^2X(x)}{dx^2}\Theta(t)$$

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

$$\frac{\partial Y}{\partial t} = \alpha\left(\frac{\partial^2 Y}{\partial x^2}\right)$$

Substituting:

$$X(x)\frac{d\Theta(t)}{dt} = \alpha\frac{d^2X(x)}{dx^2}\Theta(t)$$

$$\underbrace{\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt}}_{\text{function of time only}} = \underbrace{\alpha\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}}_{\text{function of position } (x) \text{ only}} \quad \Rightarrow \quad \underset{\text{constant}}{= \lambda}$$

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

Separates into two ordinary differential equations:

$$\boxed{\begin{gathered}\frac{1}{\Theta(t)}\frac{d\Theta(t)}{dt}=\lambda\\ \alpha\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=\lambda\end{gathered}}$$

Solve.

Apply BCs.

Apply ICs.

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Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

$$\left(\frac{T_1-T}{T_1-T_o}\right)=\frac{4}{\pi}\left\{e^{\frac{-\pi^2\alpha t}{4H^2}}\sin\frac{\pi x}{2H}+\frac{1}{3}e^{\frac{-3^2\pi^2\alpha t}{4H^2}}\sin\frac{3\pi x}{2H}+\frac{1}{5}e^{\frac{-5^2\pi^2\alpha t}{4H^2}}\sin\frac{5\pi x}{2H}+\cdots\right\}$$

Geankoplis 4th ed., eqn 5.3-6, p363

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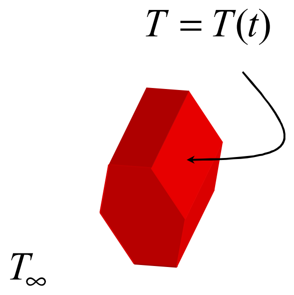
Unsteady Macroscopic Energy Balances

Example 3: If a piece of steel at $T = T_o$ is dropped into a large reservoir of fluid at T_∞, what is the temperature of the steel as a function of time?



k = large, which means that there is no resistance to heat transfer in the steel

Therefore, we are NOT calculating a temperature profile

Use Macroscopic Energy Balance

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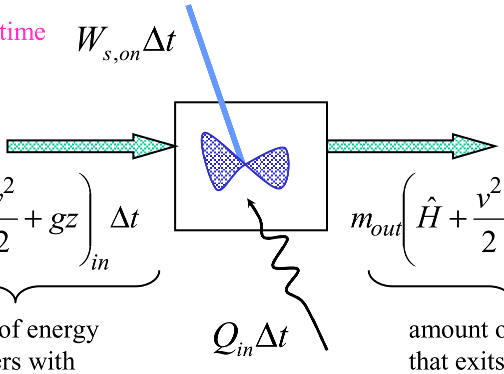
Unsteady Macroscopic Energy Balances

see Felder and Rousseau or Himmelblau

balance over time interval Δt



$$\underbrace{m_{in}\left(\hat{H}+\frac{v^2}{2}+gz\right)_{in}\Delta t}$$

amount of energy that enters with the flow between t and $t+\Delta t$

$$\underbrace{m_{out}\left(\hat{H}+\frac{v^2}{2}+gz\right)_{out}\Delta t}$$

amount of energy that exits with the flow between t and $t+\Delta t$

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Unsteady Macroscopic Energy Balances

$$accumulation = input - output$$

$$\frac{d\left(U_{sys} + \cancel{E_{k,sys}} + \cancel{E_{p,sys}}\right)}{dt} = -\Delta H - \Delta E_p - \Delta E_k + Q_{in} + \cancel{W_{s,on}}$$

Our case, no shafts

For negligible changes in E_p and E_k,
and no phase change,
and single-input, single-output system,

$$\frac{dU_{sys}}{dt} = M_{sys} C_v \frac{dT_{sys}}{dt}$$

$$\Delta H = \cancel{m} C_p \left(T_{out} - T_{in}\right)$$

Our case, no flow

Unsteady Macroscopic Energy Balances

$$M_{sys} C_v \frac{dT_{sys}}{dt} = \cancel{m} C_p \left(T_{in} - T_{sys}\right) + Q_{in} + \cancel{W_{s,on}}$$

Our case, no flow

Our case, no shafts

- •For negligible changes in E_p and E_k
- •and no phase change
- •and single-input, single-output system
- •assume $T_{out} = T_{sys}$

Unsteady Macroscopic Energy Balance Applied to cooling steel part:

$$\underbrace{M_{sys}C_v\frac{dT_{sys}}{dt}}=Q_{in}$$

The temperature
changes in the slug are
due to the heat loss

The heat loss depends on the
heat-transfer coefficient from
the slug to the environment

$$\boxed{Q_{in}=hA(T-T_{\infty})}$$