10/24/07

3. Flow down an inclined plane with

$$\mu = \mu_0 e^{-\alpha\left(\frac{x}{H}\right)}$$

The soln is the same as in class for τ_{xz}:

$$\tau_{xz} = \rho g \cos\beta\, x = -\mu \frac{dv_z}{dx}$$

$$\left(\mu = \mu_0 e^{-\alpha\left(\frac{x}{H}\right)}\right)$$

solve for v_z:

$$-\mu_0 e^{-\alpha \frac{x}{H}} \frac{dv_z}{dx} = (\rho g \cos\beta)\, x$$

$$\frac{dv_z}{dx} = \underbrace{\left(\frac{\rho g \cos\beta}{-\mu_0}\right)}_{\equiv B} x\, e^{+\frac{\alpha x}{H}}$$

$$\frac{dv_z}{dx} = B\, x\, e^{\frac{\alpha}{H} x}$$

use calculator
or
look up in Table
or
integrate by parts

Integration by parts:

$$\int u\,dv = uv - \int v\,du + C_1$$

$$u = x \qquad dv = e^{\frac{\alpha}{H}x}\,dx$$

$$du = dx \qquad v = \frac{H}{\alpha} e^{(\alpha/H)x}$$

$$\int x e^{\frac{\alpha}{H}x}\,dx = \frac{H}{\alpha} x e^{(\alpha/H)x} - \int \frac{H}{\alpha} e^{\frac{\alpha}{H}x}\,dx + C_1$$

$$= \frac{H}{\alpha} x e^{\frac{\alpha}{H}x} - \frac{H^2}{\alpha^2} e^{\frac{\alpha}{H}x} + C_1$$

Back to our calculations:

$$\frac{dv_z}{dx} = Bxe^{\alpha/H\,x}$$

integration constant

$$V_z = B\left[\frac{H}{\alpha} x e^{\frac{\alpha}{H}x} - \frac{H^2}{\alpha^2} e^{\frac{\alpha}{H}x} + C_1\right]$$

$$\boxed{\text{Boundary condition}: \quad x = H, \quad v_z = 0}$$

$$0 = \cancel{B}\left[\frac{H}{\alpha} H e^{\alpha} - \frac{H^2}{\alpha^2} e^{\alpha} + C_1\right]$$

$$\Rightarrow C_1 = \frac{H^2}{\alpha^2} e^{\alpha} - \frac{H^2}{\alpha} e^{\alpha}$$

$$= \frac{H^2}{\alpha^2} e^{\alpha}\left[1 - \alpha\right]$$

substituting back:

$$\frac{v_z}{B} = \frac{H}{\alpha} x e^{\frac{\alpha}{H}x} - \frac{H^2}{\alpha^2} e^{\frac{\alpha}{H}x} + \frac{H^2}{\alpha^2} e^{\alpha}(1-\alpha)$$

$$= \frac{H^2}{\alpha^2} e^{\frac{\alpha}{H}x}\left(\frac{\alpha x}{H} - 1\right) + \frac{H^2}{\alpha^2} e^{\alpha}(1-\alpha)$$

$$\boxed{v_z = \frac{H^2}{\alpha^2}\left(\frac{-\rho g \cos\beta}{\mu_0}\right)\left[e^{\frac{\alpha}{H}x}\left(\frac{\alpha x}{H} - 1\right) + e^{\alpha}(1-\alpha)\right]}$$

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