General Energy Transport Equation
(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, \( V \), enclosed by a surface, \( S \).

Gibbs notation:
\[
\rho \hat{c}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S_e
\]

see handout for component notation
General Energy Transport Equation

(microscopic energy balance)

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S_e \]

- **Convection**
  - Rate of change
  - Velocity must satisfy equation of motion, equation of continuity
  - See handout for component notation

- **Conduction** (all directions)
  - Energy generated per unit volume per time

The **Equation of Energy** for systems with constant \( k \)

**Microscopic energy balance, constant thermal conductivity; Gibbs notation**

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S_e \]

**Microscopic energy balance, constant thermal conductivity; Cartesian coordinates**

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v}_x \frac{\partial T}{\partial x} + \mathbf{v}_y \frac{\partial T}{\partial y} + \mathbf{v}_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e \]

**Microscopic energy balance, constant thermal conductivity; cylindrical coordinates**

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v}_r \frac{\partial T}{\partial r} + \mathbf{v}_\theta \frac{\partial T}{\partial \theta} + \mathbf{v}_z \frac{\partial T}{\partial z} \right) = k \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + S_e \]

**Microscopic energy balance, constant thermal conductivity; spherical coordinates**

\[ \rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v}_r \frac{\partial T}{\partial r} + \mathbf{v}_\theta \frac{\partial T}{\partial \theta} + \mathbf{v}_\phi \frac{\partial T}{\partial \phi} \right) = k \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + S_e \]
Example 3: Heat flux in a cylindrical shell – Temp BC

Assumptions:
• long pipe
• steady state
• \( k \) = thermal conductivity of wall

What is the steady state temperature profile in a cylindrical shell (pipe) if the inner wall is at \( T_1 \) and the outer wall is at \( T_2 \)? (\( T_1 > T_2 \))

Let’s try.
Example 3: Heat flux in a cylindrical shell – Temp BC

Solution:

\[
q_r = \frac{c_1}{A} r
\]

\[
T = -\frac{c_1}{k} \ln r + c_2
\]

Boundary conditions?

Solution for Cylindrical Shell:

\[
\frac{q_r}{A} = \frac{T_1 - T_2}{\ln \frac{R_2}{R_1}} \left( \frac{k}{r} \right)
\]

The heat flux \(\frac{q_r}{A}\) **DOES** depend on, \(k\); also \(\frac{q_r}{A}\) decreases as \(1/r\)

\[
\frac{T_2 - T}{T_2 - T_1} = \frac{\ln \frac{R_2}{R_1}}{\ln \frac{R_2}{R_1}}
\]

Note that \(T(r)\) does not depend on the thermal conductivity, \(k\) (steady state)

Pipe with temperature BCs
Example 3: Heat flux in a cylindrical shell – Temp BC

Dimensionless Temperature Profile in a pipe; $R_1=1$, $R_2=2$

\[ \frac{T_2 - T}{T_2 - T_1} \]

Pipe with temperature BCs

Example 4: Heat flux in a cylindrical shell – Newton’s law of cooling

Assumptions:
• long pipe
• steady state
• $k$ = thermal conductivity of wall
• $h_1, h_2$ = heat transfer coefficients

What is the steady state temperature profile in a cylindrical shell (pipe) if the fluid on the inside is at $T_{b1}$ and the fluid on the outside is at $T_{b2}$? ($T_{b1}>T_{b2}$)
Example 4: Heat flux in a cylindrical shell

You try.

Solution:
\[ \frac{q_r}{A} = \frac{c_1}{r} \]
\[ T = -\frac{c_1}{k} \ln r + c_2 \]

Boundary conditions?

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Example 4: Heat flux in a cylindrical shell

\[
\begin{align*}
\frac{c_1}{R_1} &= h_1(T_{b1} - T_{w1}) \\
\frac{c_1}{R_2} &= h_2(T_{w2} - T_{b2}) \\
T_{w1} &= -\frac{c_1}{k} \ln R_1 + c_2 \\
T_{w2} &= -\frac{c_1}{k} \ln R_2 + c_2
\end{align*}
\]

4 equations
4 unknowns: \( c_1, T_{w1}, c_2, T_{w2} \)

Example 4: Heat flux in a cylindrical shell

Solution: Radial Heat Flux in an Annulus

\[
T - T_{b2} = \frac{(T_{b1} - T_{b2}) \left( \ln \frac{R_2}{R} + \frac{k}{h_2 R_2} \right)}{\frac{k}{h_2 R_2} + \ln \frac{R_2}{R_1} + \frac{k}{h_1 R_1}}
\]

\[
\frac{q_r}{A} = \frac{1}{\frac{1}{h_2 R_2} + \frac{1}{k} \ln \frac{R_2}{R_1} + \frac{1}{h_1 R_1}} \left( \frac{1}{r} \right)
\]
Example 5: Heat Conduction with Generation

What is the steady state temperature profile in a wire if heat is generated uniformly throughout the wire at a rate of \( S_e \) W/m\(^3 \) and the outer radius is held at \( T_w \)?

\[ S_e = \text{energy production per unit volume} \]

You try.
Compare solutions

\[ T - T_i = \frac{1}{\ln \frac{R_2}{R_1}} \ln \frac{r}{R_1} \]

\[ T - T_w = \frac{1}{S \cdot R^2 / 4k} \left( \frac{r}{R} \right)^2 \]

Example 6: Wall heating of laminar flow. What is the steady state temperature profile in a flowing fluid in a tube if the walls are heated (constant flux, \(q_1/A\)) and if the fluid is a Newtonian fluid in laminar flow?

assume:
constant viscosity

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Example 5: Wall heating of laminar flow

We need to solve this partial differential equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( \frac{\partial T}{\partial z} \right) - \frac{\rho C_p}{k} v_z(r) \frac{\partial T}{\partial t} = 0$$

with the appropriate boundary conditions. To see the solution see:

SUMMARY

Steady State Heat Transfer

Example 1: Heat flux in a rectangular solid – Temperature BC
Example 2: Heat flux in a rectangular solid – Newton’s law of cooling
Example 3: Heat flux in a cylindrical shell – Temperature BC
Example 4: Heat flux in a cylindrical shell – Newton’s law of cooling
Example 5: Heat conduction with generation
Example 6: Wall heating of laminar flow

Conclusion: When we can simplify geometry, assume steady state, assume symmetry, the solutions are easily obtained
SUMMARY

**Steady State Heat Transfer**
Example 1: Heat flux in a rectangular solid – Temperature BC
Example 2: Heat flux in a rectangular solid – Newton’s law of cooling
Example 3: Heat flux in a cylindrical shell – Temperature BC
Example 4: Heat flux in a cylindrical shell – Newton’s law of cooling
Example 5: Heat conduction with generation
Example 6: Wall heating of laminar flow

**Unsteady State Heat Transfer**

???