Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature $T_0$. At time $t = 0$, the left face of the slab is exposed to an environment at temperature $T_1$. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, $k$, density, $\rho$, and heat capacity, $C_p$. 
Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature $T_0$. At time $t = 0$, the left face of the slab is exposed to an environment at temperature $T_1$. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, $k$, density, $\rho$, and heat capacity, $C_p$.

This is code for:

“Newton’s law of cooling boundary conditions:”

$$|\text{flux}| = h|T_{\text{bulk}} - T_{\text{wall}}|$$

Unsteady State Heat Transfer

Example: Unsteady Heat Conduction in a Semi-infinite solid

H, D, very large
Initial Condition:

\[
\begin{align*}
\text{t < 0} & \quad T = T_0 \\
\text{t \geq 0} & \quad T = T_1 \\
\text{t > 0} & \quad T = T(x, t)
\end{align*}
\]

General Energy Transport Equation
(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, \( V \), enclosed by a surface, \( S \).

Gibbs notation:

\[
\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S
\]

see handout for component notation

© Faith A. Morrison, Michigan Tech U.
General Energy Transport Equation

( microscopic energy balance)

\[ \rho C_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S \]

- \( \rho \): density
- \( C_p \): specific heat capacity
- \( \mathbf{v} \): velocity
- \( k \): thermal conductivity
- \( S \): source

Rate of change

Convection

Source

Conduction (all directions)

Velocity must satisfy equation of motion, equation of continuity

See handout for component notation

Equation of Energy

For Newtonian fluids of constant density, \( \rho \), and thermal conductivity, \( k \), with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).


Gibbs notation (vector notation)

\[ \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{k}{\rho C_p} \nabla^2 T + \frac{S}{\rho C_p} \]

Cartesian (xyz) coordinates:

\[ \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p} \]

Cylindrical (r\(\theta\)z) coordinates:

\[ \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p} \]

Spherical (r\(\theta\)\(\phi\)) coordinates:

\[ \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} = \frac{k}{\rho C_p} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{\partial^2 T}{\partial \phi^2} \right) + \frac{S}{\rho C_p} \]
Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature $T_0$. At time $t = 0$, the left face of the slab is exposed to an environment at temperature $T_1$. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, $k$, density, $\rho$, and heat capacity, $C_p$.

Newton’s law of cooling BC’s:

$$|q_x| = hA\left[T_{\text{bulk}} - T_{\text{surface}}\right]$$

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p}$$

$$\alpha \equiv \frac{k}{\rho C_p} = \text{thermal diffusivity}$$

what are the boundary conditions? initial conditions?
Example 7: Unsteady Heat Conduction in a Semi-infinite solid

You try.

Unsteady State Heat Conduction in a Semi-Infinite Slab

Initial condition: \( t = 0, T = T_o \ \forall \ x \)

Boundary conditions:

\[ x = 0, \quad q_x = hA(T - T_1) \quad \forall \ t > 0 \]
\[ x = \infty, \quad T = T_o \quad \forall \ t \]
Unsteady State Heat Conduction in a Semi-Infinite Slab

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right)
\]

Initial condition:
\[ t = 0, \ T = T_o \ \forall \ x \]

Boundary conditions:
\[ x = 0, \ \ \ q_x = h(A(T - T_1)) \ \ \forall \ t > 0 \]
\[ x = \infty, \ \ T = T_o \ \forall \ t \]

"for all t"

The solution is obtained by combination of variables.

\[ \beta \equiv \frac{h\sqrt{\alpha t}}{k} \]
\[ \zeta \equiv \frac{x}{2\sqrt{\alpha t}} \]

© Faith A. Morrison, Michigan Tech U.
Unsteady State Heat Conduction in a Semi-Infinite Slab

Solution:

$$\frac{T - T_0}{T_1 - T_0} = \text{erfc} \zeta - e^{\beta(2\zeta + \beta)} \text{erfc}(\zeta + \beta)$$

$$\beta \equiv \frac{h\sqrt{\alpha t}}{k} \quad \zeta \equiv \frac{x}{2\sqrt{\alpha t}}$$

To make this solution easier to use, we can plot it.

Complementary error function of $y$

$$\text{erfc}(y) \equiv 1 - \text{erf}(y)$$

Error function of $y$

$$\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-(y')^2} dy'$$

© Faith A. Morrison, Michigan Tech U.
Unsteady State Heat Conduction in a Semi-Infinite Slab

This:
\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \, \zeta - e^{\beta(2\zeta + \beta)} \, \text{erfc}(\zeta + \beta)
\]

Versus this: \( \zeta \equiv \frac{x}{2\sqrt{\alpha \, t}} \)

At various values of this:
\[
\beta \equiv \frac{h\sqrt{\alpha \, t}}{k}
\]

To make this solution easier to use, we can plot it.

Unsteady State Heat Conduction in a Semi-Infinite Slab

Geankoplis 4th ed., Figure 5.3-3, page 364

© Faith A. Morrison, Michigan Tech U.
With modern tools, we can plot the solution directly (evaluated on Mathcad)

**Unsteady State Heat Conduction in a Semi-Infinite Slab**

\[
\frac{T - T_o}{T_1 - T_o} = \begin{cases} 
1 & t = 0.1, 0.2, 0.5, 1, 5, 10, 50, 100 \\
\end{cases}
\]

\(\frac{T - T_o}{T_1 - T_o}\) increasing time, \(t\)

How could we use this solution?

**Example:** Will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 ft under ground? Use the following physical properties:

\[
h = 2.0 \frac{BTU}{h \text{ ft}^2 \ ^oF}
\]

\[
\alpha_{soil} = 0.018 \frac{ft^2}{h}
\]

\[
k_{soil} = 0.5 \frac{BTU}{h \text{ ft} \ ^oF}
\]
Unsteady State Heat Conduction in a Semi-Infinite Slab

\[ \frac{T - T_0}{T_1 - T_0} = \text{erfc} \, \zeta - e^{\beta (2 \zeta + \beta)} \, \text{erfc}(\zeta + \beta) \]

Both \( \zeta \) and \( \beta \) depend on time

\[ \zeta \equiv \frac{x}{2\sqrt{\alpha \, t}} \]
\[ \beta \equiv \frac{k}{\sqrt{\alpha \, t}} \]

\( T_0 = ? \)
\( T_1 = ? \)
\( T = ? \)

\[ \frac{T - T_0}{T_1 - T_0} = ? \]

Geankoplis 4th ed., Figure 5.3-3, page 364

© Faith A. Morrison, Michigan Tech U.
Unsteady State Heat Conduction in a Semi-Infinite Slab

\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \zeta - e^{\beta(2\zeta + \beta)} \text{erfc}(\zeta + \beta)
\]

Answer:
\[ t = 480 \text{ hours} \approx 20 \text{ days} \]

Example 8: Unsteady Heat Conduction in a Finite-sized solid

- The slab is tall and wide, but of thickness 2H
- Initially at \( T_0 \)
- At time \( t = 0 \) the temperature of the sides is changed to \( T_1 \)
Unsteady State Heat Transfer

Use same microscopic energy balance eqn as before.

\[
\rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S
\]

Microscopic Energy Equation in Cartesian Coordinates

\[
\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p}
\]

\[
\alpha = \frac{k}{\rho \hat{C}_p} = \text{thermal diffusivity}
\]

what are the boundary conditions? initial conditions?
Example 8: Unsteady Heat Conduction in a Finite-sized solid

You try.

Unsteady State Heat Conduction in a Finite Slab

\[ \frac{\partial T}{\partial t} = \frac{k}{\rho \bar{C}_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right) \]

Initial condition: \( t = 0, T = T_o \ \forall \ x \)

Boundary conditions:
\[
\begin{align*}
    x &= 0, & T &= T_1 \\
    x &= 2H, & T &= T_1
\end{align*}
\]
\[ \forall \ t > 0 \]
Q: How can two completely different situations give the same governing equation?
A: The boundary conditions make all the difference

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} \right)\]

\[
\text{Initial condition: } t = 0, T = T_0 \forall x
\]

\[
\text{Boundary conditions: } \begin{cases} x = 0, & T = T_1 \\ x = 2H, & T = T_1 \end{cases} \forall t > 0
\]

For more solutions to this equation see Carslaw and Jeager, Conduction of Heat in Solids, 2nd Edition, Oxford, 1959

© Faith A. Morrison, Michigan Tech U.

Unsteady State Heat Conduction in a Finite Slab

The solution is obtained by separation of variables.
Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

Let \( Y = \left( \frac{T_i - T}{T_j - T_0} \right) = \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right) \)

Guess: \( Y = X(x)\Theta(t) \)

Initial condition:
\[ t = 0, \, T = T_o \quad \forall \, x \Rightarrow Y = 1 \]

Boundary conditions:
\[ \begin{align*}
  x = 0, & \quad T = T_1 \Rightarrow Y = 0 \\
  x = 2H, & \quad T = T_1 \Rightarrow Y = 0 \\
\end{align*} \quad \forall \, t > 0 \]

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

\[ Y = X(x)\Theta(t) \quad \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right) \]

\[ \begin{align*}
  \frac{\partial Y}{\partial t} &= \frac{\partial}{\partial t} (X(x)\Theta(t)) = X(x)\frac{d\Theta(t)}{dt} \\
  \frac{\partial Y}{\partial x} &= \frac{\partial}{\partial x} (X(x)\Theta(t)) = \frac{dX(x)}{dx} \Theta(t) \\
  \frac{\partial^2 Y}{\partial x^2} &= \frac{d^2 X(x)}{dx^2} \Theta(t) \\
\end{align*} \]

© Faith A. Morrison, Michigan Tech U.
Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

\[ \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right) \]

Substituting:

\[ X(x) \frac{d\Theta(t)}{dt} = \alpha \frac{d^2 X(x)}{dx^2} \Theta(t) \]

\[ \frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} \]

\[ \Rightarrow = \lambda \]

The function of two variables is separable into two functions of one variable.

Separates into two ordinary differential equations:

\[ \frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \lambda \]

\[ \alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \lambda \]

Solve.

Apply BCs.

Apply ICs.
Temperature Profile for Unsteady State Heat Conduction in a Finite Slab

\[
\left( \frac{T_1 - T}{T_1 - T_0} \right) = \frac{4}{\pi} \left\{ \frac{-\pi^2 \alpha t}{4H^2} \sin \frac{\pi x}{2H} + \frac{1}{3} \frac{-3\pi^2 \alpha t}{4H^2} \sin \frac{3\pi x}{2H} \right. \\
\left. + \frac{1}{5} \frac{-5\pi^2 \alpha t}{4H^2} \sin \frac{5\pi x}{2H} + \cdots \right\}
\]

Geankoplis 4th ed., eqn 5.3-6, p363

Microscopic Energy Balance – is the correct physics for many problems!

Tricky step:

solving for \( T \) field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

**Strategy:**

- Look up solution in literature
- solve using numerical methods
  (e.g. Comsol)

**** Or ****

- Develop correlations on complex systems by using Dimensional Analysis
Fluid Mechanics: What did we do?

- Turbulent tube flow
- Noncircular conduits
- Drag on obstacles

1. Find a simple problem that allows us to identify the physics
2. Nondimensionalize
3. Explore that problem
4. Take data and correlate
5. Solve real problems

Solve. Real. Problems.

Powerful.

Works on heat transfer too.

More Complex Heat Transfer – Dimensional Analysis

Professor Faith Morrison
Department of Chemical Engineering
Michigan Technological University

© Faith A. Morrison, Michigan Tech U.