Examples of (simple) Heat Conduction

Cooler fluid at $T_{S2}$
Hot fluid at $T_{S1}$

$S_{w}$ - energy production per unit volume

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But these are highly simplified geometries

How do we handle complex geometries, complex flows, complex machinery?
**Engineering Modeling**

- Choose an idealized problem and solve it
- From insight obtained from ideal problem, identify governing equations of real problem
- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
- Design experiments to test modeling thus far
- Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
- Design additional experiments
- Iterate until useful correlations result

**Experience with Dimensional Analysis thus far:**

- Flow in pipes at all flow rates (laminar and turbulent)
  *Solution:* Navier-Stokes, Re, Fr, L/D, dimensionless wall force = \( f; f = f(Re, L/D) \)

- Rough pipes
  *Solution:* add additional length scale; then nondimensionalize

- Non-circular conduits
  *Solution:* Use hydraulic diameter as the length scale of the flow to nondimensionalize

- Flow around obstacles (spheres, other complex shapes)
  *Solution:* Navier-Stokes, Re, dimensionless drag = \( CD; CD = CD(Re) \)

- Boundary layers
  *Solution:* Two components of velocity need independent lengthscales

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These have been exhilarating victories for dimensional analysis.
Now, move to heat transfer:

- Forced convection heat transfer from fluid to wall  
  Solution: ?

- Natural convection heat transfer from fluid to wall  
  Solution: ?

- Radiation heat transfer from solid to fluid  
  Solution: ?

We have already started using the results/techniques of dimensional analysis through defining the heat transfer coefficient, \( h \).
Now, move to heat transfer:

- Forced convection heat transfer from fluid to wall
  \[ h = f(Re) \]

- Natural convection heat transfer from fluid to wall
  \[ h = \text{[solution]} \]

- Radiation heat transfer from solid to fluid
  \[ h = \text{[solution]} \]

We have already started using the results/techniques of dimensional analysis through defining the heat transfer coefficient, \( h \) (recall that we did this in fluids too: we used \( f(Re) \) long before we knew where that all came from)

Handy tool: Heat Transfer Coefficient

Temperature variation near-wall region is caused by complex phenomena that are lumped together into the heat transfer coefficient, \( h \)
The flux at the wall is given by the empirical expression known as Newton’s Law of Cooling.

This expression serves as the definition of the heat transfer coefficient.

\[ \frac{q_x}{A} \equiv h \left| T_{bulk} - T_{wall} \right| \]

\( h \) depends on:
- geometry
- fluid velocity
- fluid properties
- temperature difference

To get values of \( h \) for various situations, we need to measure data and create data correlations (dimensional analysis).
Complex Heat transfer Problems to Solve:
• Forced convection heat transfer from fluid to wall
  Solution: ?

• Natural convection heat transfer from fluid to wall
  Solution: ?

• Radiation heat transfer from solid to fluid
  Solution: ?

• The values of $h$ will be different for these three situations (different physics)
• Investigate simple problems in each category, model them, take data, correlate

Chosen problem: Forced Convection Heat Transfer
Solution: Dimensional Analysis

Following procedure familiar from pipe flow,
• What are governing equations?
• Scale factors (dimensionless numbers)?
• Quantity of interest?
  Heat flux at the wall
General Energy Transport Equation
(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, $V$, enclosed by a surface, $S$.

\[ \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v \cdot \nabla T \right) = k \nabla^2 T + S \]

Gibbs notation:

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**Equation of energy** for Newtonian fluids of constant density, \( \rho \), and thermal conductivity, \( k \), with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

\[
\left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{k}{\rho C_p} \nabla^2 T + \frac{S}{\rho C_p}
\]

Gibbs notation (vector notation)

\[
\left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{k}{\rho C_p} \nabla^2 T + \frac{S}{\rho C_p}
\]

Cartesian (xyz) coordinates:

\[
\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial x} + v_\theta \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p}
\]

Cylindrical (r\( \theta \zeta \)) coordinates:

\[
\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial T}{\partial z} \right) + \frac{S}{\rho C_p}
\]

Spherical (r\( \theta \phi \)) coordinates:

\[
\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} = \frac{k}{\rho C_p} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \right) + \frac{S}{\rho C_p}
\]

**Example: Heat flux in a cylindrical shell**

**Assumptions:**
- Long pipe
- Steady state
- \( k \) = thermal conductivity of wall
- \( h_1, h_2 \) = heat transfer coefficients

What is the steady state temperature profile in a cylindrical shell (pipe) if the fluid on the inside is at \( T_{b1} \) and the fluid on the outside is at \( T_{b2} \) (\( T_{b1} > T_{b2} \))?
Consider: Heat-transfer to/from flowing fluid inside of a tube – forced-convection heat transfer

\[ T_1 = \text{core bulk temperature} \]
\[ T_0 = \text{wall temperature} \]
\[ T(r, \theta, z) = \text{temp distribution in the fluid} \]

In principle, with the right math/computer tools, we could calculate the complete temperature and velocity profiles in the moving fluid.

**What are governing equations?**
Microscopic energy balance plus Navier-Stokes, continuity

**Scale factors?**
Re, Fr, L/D plus whatever comes from the rest of the analysis

**Quantity of interest (like wall force, drag)?**
Heat transfer coefficient

The quantity of interest in forced-convection heat transfer is \( h \)

How is the heat transfer coefficient related to the full solution for \( T(r, \theta, z) \)?
At the boundary, (Newton's Law of Cooling is the **boundary condition**)

\[
\frac{q_r}{A} = h|T_1 - T_0|
\]

\[
Q = (2\pi RL)(h)(T_1 - T_0)
\]

We can calculate the total heat transferred from \(T(r, \theta, z)\) in the fluid:

\[
Q = \int \int [\hat{n} \cdot \bar{q}]_{\text{surface}} dS
\]

We need \(T(r, \theta, z)\) in the fluid.
Equate these two: Total heat flow through the wall

\[ (2\pi RL)(h)(T_1 - T_0) = \int_S \left[ \hat{e}_r \cdot \vec{q} \right]_{\text{surface}} dS \]

Total heat flow through the wall in terms of \( h \)

\[ (2\pi RL)(h)(T_1 - T_0) = Q = \int_0^2 \int_0^L -k \frac{\partial T}{\partial r} \bigg|_{r=R} \ R dz d\theta \]

Total heat conducted to the wall from the fluid

Now, non-dimensionalize this expression
Non-dimensionalize

Non-dimensional variables:

**Position:**
- \( r^* = \frac{r}{D} \)
- \( z^* = \frac{z}{D} \)

**Temperature:**
- \( T^* = \frac{T - T_0}{(T_1 - T_0)} \)

\[ h(\pi DL)(T_1 - T_0) = \int_0^{2\pi} \int_0^{L/D} \left( \frac{\partial T^*}{\partial r^*} \right)_{r^*=1/2} - k \frac{\partial T^*}{\partial r^*} \left( \frac{T_1 - T_0}{D} \right) \frac{D^2}{2} dz^* d\theta \]

\[ 2\pi \left( \frac{hD}{k} \right) \left( \frac{L}{D} \right) = \int_0^{2\pi} \int_0^{L/D} \left( \frac{\partial T^*}{\partial r^*} \right)_{r^*=1/2} dz^* d\theta \]

**Nusselt number, Nu**
- (dimensionless heat-transfer coefficient)

\[ Nu = Nu \left( T^*, \frac{L}{D} \right) \]

one additional dimensionless group

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h(DL)(T_1 - T_0) = \int_0^{L/D} \int_0^D -k \frac{\partial T^*}{\partial r^*} \, dr^* \, dz^* d\theta

\frac{2\pi hD}{k} \left( \frac{L}{D} \right) = \int_0^{L/D} \int_0^D -\frac{\partial T^*}{\partial r^*} \, dr^* \, dz^* d\theta

Nusselt number, Nu
(dimensionless heat-transfer coefficient)

\frac{Nu}{Nu} = Nu\left( \frac{T^*}{L}, \frac{D}{D} \right)

This is a function of Re through \nu

Non-dimensional Energy Equation

\frac{\partial T^*}{\partial t^*} + v_x^* \frac{\partial T^*}{\partial x^*} + v_y^* \frac{\partial T^*}{\partial y^*} + v_z^* \frac{\partial T^*}{\partial z^*} = \frac{1}{Pe} \left( \frac{\partial^2 T^*}{\partial x^*} + \frac{\partial^2 T^*}{\partial y^*} + \frac{\partial^2 T^*}{\partial z^*} \right) + S^*

Non-dimensional Navier-Stokes Equation

\frac{Dv_z^*}{Dt} = -\frac{\partial P^*}{\partial z^*} + \frac{1}{Re} \frac{\partial}{\partial x^*} \left( \nabla^2 v_z^* \right) + \frac{1}{Pr} \frac{\partial}{\partial y^*} g^*

Non-dimensional Continuity Equation

\frac{\partial v_x^*}{\partial x^*} + \frac{\partial v_y^*}{\partial y^*} + \frac{\partial v_z^*}{\partial z^*} = 0

Quantity of interest

\frac{Nu}{Nu} = \frac{1}{2\pi L/D} \int_0^{L/D} -\frac{\partial T^*}{\partial r^*} \, dr^* \, dz^* d\theta
According to our **dimensional analysis** calculations, the dimensionless heat transfer coefficient should be found to be a function of four dimensionless groups:

\[
Nu = Nu \left( \frac{Re}{Pr, Pr, \frac{L}{D}} \right)
\]

**Peclet number**
\[
P_e \equiv \frac{\rho \dot{C}_p \nu D}{k} = \frac{\dot{C}_p \mu \nu D}{k}
\]

**Prandtl number**
\[
Pr \equiv \frac{\dot{C}_p \mu}{k}
\]

Now, do the experiments.

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**Forced Convection Heat Transfer**

- Build apparatus (*several* actually, with different D, L)
- Run fluid through the inside (at different V; for different fluids \( \rho, \mu, \dot{C}_p, k \))
- Measure \( T_{bulk} \) on inside; \( T_{wall} \) on inside
- Measure rate of heat transfer, \( Q \)
- Calculate \( h \): \( |Q| = hA|T_{bulk} - T_{wall}| \)
- Report \( h \) values in terms of dimensionless correlation:

\[
Nu = \frac{hD}{k} = f \left( Re, Pr, \frac{L}{D} \right)
\]

It should only be a function of these dimensionless numbers (**if** our Dimensional Analysis is correct……)
Correlations for Forced Convection Heat Transfer Coefficients

\[ \text{Nu} = 0.027 \text{Re}^{0.8} \text{Pr}^{0.14} \left( \frac{\mu_b}{\mu_w} \right) \]

\[ \text{Nu} = 1.86 \left( \frac{\text{RePrD}}{L} \right)^{0.14} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]

Pr = 8.07 (water, 60°F)

viscosity ratio = 1.00

L/D = 65

If dimensional analysis is right, we should get a single curve, not multiple different curves depending on: \( D, L, \mu, \text{etc.} \).

Geankoplis, 4th ed. eqn 4.5-4, page 260

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Correlations for Forced Convection Heat Transfer Coefficients

If dimensional analysis is right, we should get a single curve, not multiple different curves depending on: \( D, L, \mu, \text{etc.} \)

**Dimensional Analysis WINS AGAIN!**

Geankoplis, 4th ed. eqn 4.5-4, page 260

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Heat Transfer in Laminar flow in pipes: data correlation for forced convection heat transfer coefficients

\[
Nu_a = \frac{h_a D}{k} = 1.86 \left( \frac{D}{L} \right)^{0.3} Pr \frac{D}{L}^{0.14} \left( \frac{\mu_w}{\mu} \right)^{0.14}
\]

the subscript “a” refers to the type of average temperature used in calculating the heat flow, \( q \)

\[
q = h_a A \Delta T_a
\]

\[
\Delta T_a = \frac{(T_w - T_{bi}) + (T_w - T_{bo})}{2}
\]

Re < 2100, \( (Re Pr \frac{D}{L}) > 100 \), horizontal pipes; all physical properties evaluated at the mean temperature of the bulk fluid except \( \mu_w \) which is evaluated at the (constant) wall temperature.

Geankoplis, 4th ed. eqn 4.5-4, page 260

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Forced convection
Heat Transfer in Laminar flow in pipes

\[ \text{Nu}_a = \frac{h_a L}{k} = 1.86 \left( \text{RePr} \frac{D}{L} \right)^{\frac{1}{3}} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]

Physical Properties
evaluated at:
\[ T_{b,\text{in}} + T_{b,\text{out}} \]
\[ \frac{2}{2} \]

Forced convection
Heat Transfer in Turbulent flow in pipes

\[ \text{Nu}_L = \frac{h_l L}{k} = 0.027 \text{Re}^{0.8} \text{Pr}^{\frac{1}{3}} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]

May have to be estimated
\[ T_{b,\text{in}} + T_{b,\text{out}} \]
\[ \frac{2}{2} \]

- all physical properties (except \( \mu_w \))
evaluated at the bulk mean temperature
- Laminar or turbulent flow

\[ \text{Nu}_a = \frac{h_a D}{k} = 1.86 \left( \text{RePr} \frac{D}{L} \right)^{\frac{1}{3}} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]

(reminiscent of pipe wall roughness; needed to modify dimensional analysis to correlate on roughness)

In our dimensional analysis, we assumed constant \( \rho, k, m, \text{ etc.} \). Therefore we did not predict a viscosity-temperature dependence. If viscosity is not assumed constant, the dimensionless group shown below is predicted to appear in correlations.

\[ \text{Nu}_a = \frac{h_a D}{k} = 1.86 \left( \text{RePr} \frac{D}{L} \right)^{\frac{1}{3}} \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]
Viscous fluids with large DT

heating

lower viscosity fluid layer
speeds flow near the wall ==>
higher h

\[ \mu_b > \mu_w \]

empirical result:

\[ \left( \frac{\mu_b}{\mu_w} \right)^{0.14} \]

cooling

higher viscosity fluid layer
retards flow near the wall ==>
lower h

\[ \mu_b < \mu_w \]

Why does \( \frac{L}{D} \) appear in laminar flow correlations and not in the turbulent flow correlations?

LAMINAR

Less lateral mixing in laminar flow means more variation in \( h(z) \).
In turbulent flow, good lateral mixing reduces the variation in $h$ along the pipe length.

Example of partial solution to Homework

<table>
<thead>
<tr>
<th>Phase</th>
<th>Equation</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laminar flow in pipes</td>
<td>$Nu_{lm} = \frac{h_{lm}D}{k} = 1.86 \left( \frac{Re Pr D}{L} \right) \left( \frac{\mu_w}{\mu_{lm}} \right)^{0.34}$</td>
<td>$Re=2100$, $(RePtD/L) \approx 100$, horizontal pipes, eqn 4.5-4, page 238; all properties evaluated at the temperature of the bulk fluid except $\mu_w$, which is evaluated at the wall temperature.</td>
</tr>
<tr>
<td>Turbulent flow in smooth tubes</td>
<td>$Nu_{tm} = \frac{h_{tm}D}{k} = 0.027 Re^{0.3} Pr^{0.7} \left( \frac{\mu_w}{\mu_{tm}} \right)^{0.14}$</td>
<td>$Re=6000$, $0.7 &lt; Pr &lt; 16000$, $L/D &gt; 60$, eqn 4.5-8, page 239; all properties evaluated at the mean temperature of the bulk fluid except $\mu_w$, which is evaluated at the wall temperature. The mean is the average of the inlet and outlet bulk temperatures; not valid for liquid metals.</td>
</tr>
<tr>
<td>Air at 1 atm in turbulent flow in pipes</td>
<td>$h_{in} = \frac{3.52 \nu/(m/s)^{0.8}}{D(m)^{0.6}}$</td>
<td>equation 4.5-9, page 239</td>
</tr>
<tr>
<td></td>
<td>$h_{in} = \frac{0.5 \nu/(\ell/s)^{0.6}}{D(\ell)^{0.6}}$</td>
<td></td>
</tr>
<tr>
<td>Water in turbulent flow in pipes</td>
<td>$h_{in} = 1429\left[1 + 0.0146\left(\frac{T}{C} \right) \nu/(m/s)^{0.8} D(m)^{0.6} \right]$</td>
<td>$4 &lt; T/(C) &lt; 105$, equation 4.5-10, page 239</td>
</tr>
<tr>
<td></td>
<td>$h_{in} = 150\left[1 + 0.011\left(\frac{T}{F} \right) \nu/(\ell/s)^{0.6} D(\ell)^{0.6} \right]$</td>
<td></td>
</tr>
</tbody>
</table>
Complex Heat transfer Problems to Solve:

- Forced convection heat transfer from fluid to wall
  
  **Solution:**?

- Natural convection heat transfer from fluid to wall
  
  **Solution:**?

- Radiation heat transfer from solid to fluid
  
  **Solution:**?

We started with a forced-convection pipe problem, did dimensional analysis, and found the dimensionless numbers.

To do a situation with different physics, we must start with a different starting problem.

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Free Convection  i.e. hot air rises

- heat moves from hot surface to cold air (fluid) by radiation and conduction
- increase in fluid temperature decreases fluid density
- recirculation flow begins
- recirculation adds to the heat-transfer from conduction and radiation

⇒ coupled heat and momentum transport
Free Convection

i.e. hot air rises

How can we solve real problems involving free (natural) convection?

We'll try this: Let's review how we approached solving real problems in earlier cases, i.e. in fluid mechanics, forced convection.

Engineering Modeling

• Choose an idealized problem and solve it
• From insight obtained from ideal problem, identify governing equations of real problem
• Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
• Design experiments to test modeling thus far
• Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
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• Iterate until useful correlations result

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Example: Free convection between long parallel plates or heat transfer through double-pane glass windows

- **assumptions:**
  - long, wide slit
  - steady state
  - no source terms
  - viscosity constant
  - density varies with $T$

Calculate: $T, v$ profiles

Example: Natural convection between vertical plates

**Mass balance:**
\[ \frac{\partial \rho}{\partial t} + \rho \nabla p + \rho (\nabla \cdot v) = 0 \]

**Momentum balance:**
\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \mu \nabla^2 v + \rho g \]
Example: \textbf{Natural convection between vertical plates}

You try.

\begin{align*}
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho (\nabla \cdot \mathbf{v}) &= 0 \\
\frac{\partial \rho}{\partial t} + (\nu_x \frac{\partial \rho}{\partial x} + \nu_y \frac{\partial \rho}{\partial y} + \nu_z \frac{\partial \rho}{\partial z}) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) &= 0
\end{align*}
Mass balance:
\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho (\nabla \cdot \mathbf{v}) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0
\]

steady \( \mathbf{v} = v_z(y)\hat{e}_z \)  

Conclusion: density must not vary with \( z \).

\[
\rho = \rho(x, y)
\]

\[
\rho = \rho(y)
\]
Momentum balance:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g \]

\[ \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} + \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \]

\[ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} + \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \]

\[ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} + \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \]

Is Pressure a function of z?

YES, there should be hydrostatic pressure (due to weight of fluid)

"Pressure at the bottom of a column of fluid = pressure at top + \rho gh."

\[ p_0 = p(z) + \bar{\rho} gz \]

\[ p(z) = p_0 - \bar{\rho} gz \]

\[ \Rightarrow \frac{dP}{dz} = -\bar{\rho} g \]
To account for the temperature variation of $\rho$:

$$\rho = \bar{\rho} - \bar{\rho} \beta (T - \bar{T})$$

- $\bar{\rho}$ = mean density
- $\beta$ = volumetric coefficient of expansion at $\bar{T}$
- $\bar{T} = \frac{T_1 + T_2}{2}$

Energy balance:

$$\rho C_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S$$

$$\rho C_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S$$

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Energy balance:
\[ \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S \]
\[ \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S \]
\[
T(y) = \frac{T_1 - T_2}{2b} y + \frac{T_1 + T_2}{2}
\]
\[
T(y) = \frac{T_1 - T_2}{2b} y + T
\]

\[
\rho = \bar{\rho} - \bar{\rho} \beta (T - T_m)
\]
\[
\rho = \bar{\rho} - \bar{\rho} \beta \left( \frac{T_1 - T_2}{2b} y \right)
\]
Energy balance:
\[
\rho c_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S
\]

\[
\rho c_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S
\]

\[
T(y) = \frac{T_1 - T_2}{2b} y + \frac{T_1 + T_2}{2}
\]

\[
\rho = \bar{\rho} - \bar{\rho} \beta (T - \bar{T})
\]

\[
\rho = \bar{\rho} - \bar{\rho \beta} \left( \frac{T_1 - T_2}{2b} y \right)
\]

**Final Result:** (free convection between two slabs)

\[
v_z(y) = \frac{\bar{\rho} \beta g (T_2 - T_1) b^2}{12 \mu} \left[ \left( \frac{y}{b} \right)^3 - \left( \frac{y}{b} \right) \right]
\]
Velocity profile for free convection between two wide, tall, parallel plates

(Note that the temperature maxima are not centered)

Engineering Modeling

- Choose an idealized problem and solve it
  - From insight obtained from ideal problem, identify governing equations of real problem
  - Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
  - Design experiments to test modeling thus far
  - Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness lengthscale)
  - Design additional experiments
  - Iterate until useful correlations result

Free Convection i.e. hot air rises
Energy balance:

\[ \frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0 \]

Momentum balance:

\[ \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \]

Energy balance:

\[ \rho c_p \left( \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + S \]

**Engineering Modeling**

- Choose an idealized problem and solve it
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- Iterate until useful correlations result
Nondimensionalize the governing equations; deduce dimensionless scale factors.

To nondimensionalize the Navier-Stokes for free convection problems, we follow the simple problem we just completed: $\rho = \rho(T)$, $(u_z) = 0$.

$$\rho\left(\frac{\partial\nu}{\partial t} + \nu \cdot \nabla \nu\right) = -\nabla P + \mu \nabla^2 \nu + \rho g$$

Following the previous example, how do we handle the various densities?

---

**Return to Dimensional Analysis…**

**FORCED CONVECTION**

**EXAMPLE I:** Pressure-driven flow of a Newtonian fluid in a tube:
- steady state
- well developed
- long tube

There was an average velocity used as the characteristic velocity.
z-component of the Navier-Stokes Equation:

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right)$$

$$= -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Choose:

- $D = \text{characteristic length}$
- $V = \text{characteristic velocity}$
- $D/V = \text{characteristic time}$
- $\rho V^2 = \text{characteristic pressure}$

This velocity is an imposed (forced) average velocity.

We do not have such an imposed velocity in natural convection.

Non-dimensional variables:

- time:
  $$t^* \equiv \frac{tV}{D}$$
- position:
  $$r^* \equiv \frac{r}{D} \quad z^* \equiv \frac{z}{D}$$
- velocity:
  $$v_r^* \equiv \frac{v_r}{V} \quad v_\theta^* \equiv \frac{v_\theta}{V} \quad v_z^* \equiv \frac{v_z}{V}$$
- driving force:
  $$P^* \equiv \frac{P}{\rho V^2} \quad g_z^* \equiv \frac{g_z}{g}$$
z-component of the nondimensional Navier-Stokes Equation:

\[
\frac{Dv_z^*}{Dt} = -\frac{\partial P^*}{\partial z^*} + \frac{\mu}{\rho D}\left(\nabla^2 v_z^*\right)^* + \frac{gD}{V^2} g^*
\]

\[
\left(\nabla^2 v_z^*\right)^* = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_z^*}{\partial r}\right) + \frac{1}{r^2}\frac{\partial v_r^*}{\partial \theta} + \frac{\partial v_z^*}{\partial z^*}
\]

\[
Dv_z^* = \left(\frac{\partial v_z^*}{\partial t} + v_r^* \frac{\partial v_z^*}{\partial r} + v_\theta^* \frac{\partial v_z^*}{\partial \theta} + v_z^* \frac{\partial v_z^*}{\partial z}\right)
\]

For free convection, what is the average velocity?
Answer: zero!

for forced convection we used: \( v_z^* = \frac{v_z}{V} \) \( V \equiv \langle v \rangle \)

For free convection \( \langle v \rangle = 0 \); what \( V \) should we use for free convection?

Solution: use a Reynolds-number type expression so that no characteristic velocity imposes itself (we’ll see now how that works):

\[
v_z^* = \frac{v_z}{V} = \frac{\bar{\rho} v_z D}{\mu} \Rightarrow V \equiv \frac{\rho}{D\bar{\rho}}
\]
When non-dimensionalizing the Navier-Stokes, what do I use for $\rho$? (answer from idealized problem)

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right)$$

$$= \frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \rho g$$

here we use $\bar{\rho}$ because the issue is volumetric flow rate

as before, for pressure gradient we use $-\bar{\rho}g$

here we use $\rho(T)$ because the issue is driving the flow by density differences affected by gravity

non-dimensional variables:

- time:

$$t^* \equiv \frac{t \mu}{D^2 \bar{\rho}}$$

- position:

$$r^* \equiv \frac{r}{D} \quad z^* \equiv \frac{z}{D}$$

- velocity:

$$v_z^* \equiv \frac{v_z D \bar{\rho}}{\mu} \quad v_r^* \equiv \frac{\bar{\rho} v_r D}{\mu} \quad v_\theta^* \equiv \frac{\bar{\rho} v_\theta D}{\mu}$$

- driving force:

$$T^* = \frac{T - \bar{T}}{T_2 - T}$$

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SOLUTION: $z$-component of the nondimensional Navier-Stokes Equation (free convection):

$$\frac{Dv^*_z}{Dt} = \left( \nabla^2 v_z \right)^* + \left[ \frac{gD^3 T^*}{\mu^2} \right] T^*$$

Or any appropriate characteristic $\Delta T$:

$$\left( \nabla^2 v_z \right)^* \equiv \frac{1}{r} \frac{\partial}{\partial r} \left( r^* \frac{\partial v_z^*}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z^*}{\partial \theta^2} + \frac{\partial^2 v_z^*}{\partial z^2}$$

$$\frac{Dv^*_z}{Dt} \equiv \left( \frac{\partial v_z^*}{\partial t^*} + v_r^* \frac{\partial v_z^*}{\partial r^*} + \frac{v_\theta^*}{r^*} \frac{\partial v_z^*}{\partial \theta^*} + v_z^* \frac{\partial v_z^*}{\partial z^*} \right)$$

Grashof number $Gr \equiv \frac{gD^3 \beta^2 \Delta T}{\mu^2}$

Dimensionless Equation of Motion (free convection)

$$\frac{Dv_z^*}{Dt^*} = \left( \nabla^2 v_z \right)^* + Gr T^*$$

Dimensionless Energy Equation (free convection; $Re = 1$)

$$\left( \frac{\partial T^*}{\partial t^*} + v^* \cdot \nabla^* T^* \right) = \frac{1}{Pr} \nabla^2 T^*$$

$$Nu = Nu \left( T^*, \frac{L}{D} \right) \Rightarrow Nu = Nu \left( Pr, Gr, \frac{L}{D} \right)$$
Engineering Modeling

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Free Convection
i.e. hot air rises

Done (see literature)
**Example:** Natural convection from vertical planes and cylinders

\[
\text{Nu} = \frac{hL}{k} = aGr^m \text{Pr}^m
\]

- \(a, m\) are given in Table 4.7-1, page 255 Geankoplis for several cases
- \(L\) is the height of the plate
- all physical properties evaluated at the **film temperature**, \(T_f\)

**Free convection correlations use the film temperature for calculating the physical properties**

**Literature Results:**

- Natural convection Vertical planes and cylinders
  \[
  \text{Nu} = \frac{hL}{k} = aGr^m \text{Pr}^m
  \]
  - All physical properties evaluated at the **film temperature**, \(T_f\)

- Forced convection Heat Transfer in Laminar flow in pipes
  \[
  \text{Nu}_a = \frac{h_aL}{k} = 1.86 \left( \text{RePr} \frac{D}{L} \right)^{\frac{1}{3}} \left( \frac{\mu_b}{\mu_w} \right)^{0.14}
  \]
  - All physical properties (except \(\mu_w\)) evaluated at the **bulk mean temperature**
  - (true also for turbulent flow correlation)

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Free Convection  i.e. hot air rises

Engineering Modeling

- Choose an idealized problem and solve it
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- Nondimensionalize the governing equations; deduce dimensionless scale factors (e.g. Re, Fr for fluids)
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  - Revise modeling (structure of dimensional analysis, identity of scale factors, e.g. add roughness)
  - Design additional experiments
- Iterate until useful correlations result

Success!
(Dimensional Analysis wins again)

Practice Heat-Transfer Problems:

Forced Convection
Free Convection
Practice 1: A wide, deep rectangular oven (1.0 ft tall) is used for baking loaves of bread. During the baking process the temperature of the air in the oven reaches a stable value of $100^\circ F$. The oven side-wall temperature is measured at this time to be a stable $450^\circ F$. Please estimate the heat flux from the wall per unit width.

Reference: Geankoplis Ex. 4.7-1 page 279

Practice 2: A hydrocarbon oil enters a pipe (0.0303 ft inner diameter; 15.0 ft long) at a flow rate of 80 lbm/h. Steam condenses on the outside of the pipe, keeping the inside pipe surface at a constant $350^\circ F$. If the temperature of the entering oil is $150^\circ F$, what is temperature of the oil at the outlet of the pipe?

**Hydrocarbon oil properties:**

- Mean heat capacity = $0.50 \frac{BTU}{lbm^\circ F}$
- Thermal conductivity = $0.083 \frac{BTU}{h ft^\circ F}$
- Viscosity =
  - $6.50 \text{ cp, } 150^\circ F$
  - $5.05 \text{ cp, } 200^\circ F$
  - $3.80 \text{ cp, } 250^\circ F$
  - $2.82 \text{ cp, } 300^\circ F$
  - $1.95 \text{ cp, } 350^\circ F$

Reference: Geankoplis Ex. 4.5-5 page 269
Practice 3: Air flows through a tube (25.4 mm inside diameter, long tube) at 7.62 m/s. Steam condenses on the outside of the tube such that the inside surface temperature of the tube is 488.7 K. If the air pressure is 206.8 kPa and the mean bulk temperature of the air is \( (T_{\text{out}} + T_{\text{in}})/2 = 477.6 \) K, what is the steady-state heat flux to the air?

Reference: Geankoplis Ex. 4.5-1 page 262

Practice 4: Hard rubber tubing (inside radius = 5.0 mm; outside radius = 20.0 mm) is used as a cooling coil in a reaction bath. Cold water is flowing rapidly inside the tubing; the inside wall temperature is 274.9 K and the outside wall temperature is 297.1 K. To keep the reaction in the bath under control, the required cooling rate is 14.65 W. What is the minimum length of tubing needed to accomplish this cooling rate? What length would be needed if the coil were copper?

*Hard rubber properties:*
- Density = 1198 kg/m³
- Thermal conductivity (0°C) = 0.151 W/mK

Reference: Geankoplis Ex. 4.2-1 page 243, but don’t do it his way—follow class methods.

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Practice 5: A cold-storage room is constructed of an inner layer of pine (thickness = 12.7 mm), a middle layer of cork board (thickness = 101.6 mm), and an outer layer of concrete (thickness = 76.2 mm). The inside wall surface temperature is 255.4 K and the outside wall surface temperature is 297.1 K. What is the heat loss per square meter through the walls and what is the temperature at the interface between the wood and the cork board?

Material properties:

Thermal conductivity pine = 0.151 \( \frac{W}{mk} \)

Thermal conductivity cork board = 0.0433 \( \frac{W}{mk} \)

Thermal conductivity concrete = 0.762 \( \frac{W}{mk} \)

Reference: Geankoplis Ex. 4.3-1 page 245, but don’t do it his way—follow class methods.

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Practice 6: A thick-walled tube (stainless steel; 0.0254 m inner diameter; 0.0508 m outer diameter; length 0.305 m) is covered with a 0.0254 m thickness of insulation. The inside-wall temperature of the pipe is 811.0 K and the outside surface temperature of the insulation is 310.8 K. What is the heat loss and the temperature at the interface between the steel and the insulation?

Material properties of stainless steel:

Thermal conductivity = 21.63 \( \frac{W}{mk} \)

Density = 7861 \( \frac{kg}{m^3} \)

Heat Capacity = 490 \( \frac{J}{kg \cdot K} \)

Material properties of insulation:

Thermal conductivity = 0.2423 \( \frac{W}{mk} \)

Reference: Geankoplis Ex. 4.3-2 page 247, but don’t do it his way—follow class methods.

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Experience with Dimensional Analysis thus far:

- Flow in pipes at all flow rates (laminar and turbulent)
  
  Solution: Navier-Stokes, Re, Fr, L/D,
  dimensionless wall force = f; f=f(Re, L/D)

- Flow around obstacles (spheres, other complex shapes)
  
  Solution: Navier-Stokes, Re,
  dimensionless drag = C_D; C_D = C_D(Re)

- Forced convection heat transfer from fluid to wall
  
  Solution: Microscopic energy, Navier-Stokes, Re, Pr, L/D,
  heat transfer coefficient = h; h = h(Re, Pr, L/D)

- Natural convection heat transfer from fluid to wall
  
  Solution: Microscopic energy, Navier-Stokes, Gr, Pr, L/D,
  heat transfer coefficient = h; h = h(Gr, Pr, L/D)

Now, move to last heat-transfer mechanism:

- Radiation heat transfer from solid to fluid?
  
  Solution: ?
Experience with Dimensional Analysis thus far:

- Flow in pipes at all flow rates (laminar and turbulent)
  
  Solution: Navier-Stokes, Re, Fr, L/D,
  dimensionless wall force = f = f(Re, L/D)

- Natural convection heat transfer from fluid to wall
  
  Solution: Microscopic energy, Navier-Stokes, Gr, Pr, L/D,
  heat transfer coefficient = h; h = h(Gr, Pr, L/D)

- Forced convection heat transfer from fluid to wall
  
  Solution: Microscopic energy, Navier-Stokes, Re, Pr, L/D,
  heat transfer coefficient = h; h = h(Re, Pr, L/D)

Actually, we’ll hold off on radiation and spend some time on heat exchangers and other practical concerns

Now, move to last heat-transfer mechanism:

- Radiation heat transfer from solid to fluid?
  
  Solution: ?

Next:

CM3110
Transport I
Part II: Heat Transfer

Applied Heat Transfer: Heat Exchanger Modeling, Sizing, and Design

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