CM3110
Transport I
Part I: Fluid Mechanics

Michigan Tech

More Complicated Flows

(Dimensional Analysis, rough pipes, hydraulic diameter, porous media)

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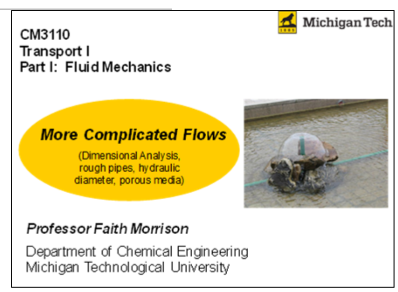
Dimensional Analysis

What's the plan?



Outline

1. Solving complicated flows with Navier-Stokes
2. *Dimensional Analysis* for
   a. Design
   b. Scale up
   c. Data correlations
3. How to Do *Dimensional Analysis*
4. What *Dimensional Analysis* looks like when it works

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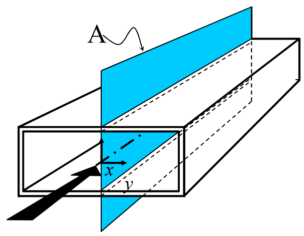
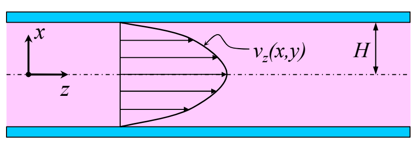
What about more complicated Newtonian problems?

EXAMPLE : Pressure-driven flow of a Newtonian fluid in a rectangular duct: **Poiseuille flow**



- steady state
- well developed
- long tube
- $P(0)=P_o$, $P(L)=P_L$

cross-section A:



$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z(x,y) \end{pmatrix}_{xyz}$$

Velocity varies in two directions

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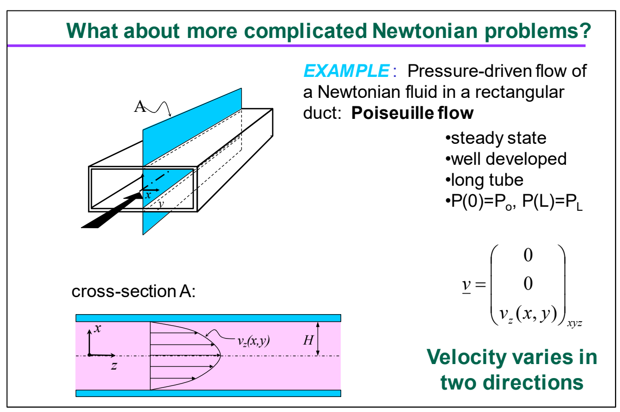
?



Let's try.

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The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z} \right) + \rho g_z$$

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www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

Navier-Stokes:

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial P}{\partial r} + \mu \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \mu \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_\theta)}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \mu \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

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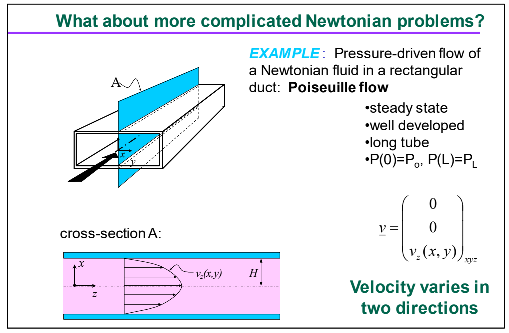
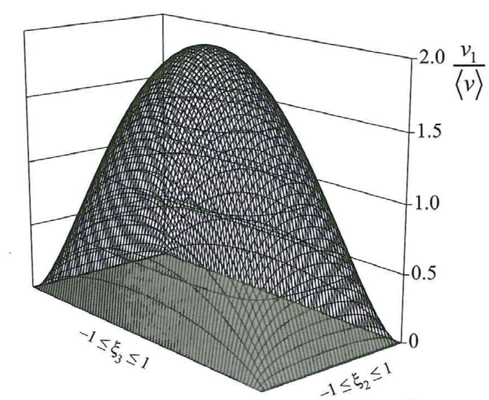
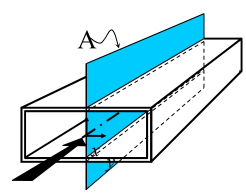
www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf



Boundary Conditions: ?



Pressure-driven flow in a rectangular duct



$$\frac{v_1(\xi_1,\xi_2)}{\langle v \rangle_{slit}} = \frac{48}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{(n-1)}{2}} \left[1 - \frac{\cosh(n\pi\mathcal{W}\xi_3/2H)}{\cosh(n\pi\mathcal{W}/2H)}\right] \cos\frac{\left(\frac{n\pi\xi_2}{2}\right)}{n^3}$$

$$\xi_2 \equiv \frac{x_2}{H} \qquad \xi_3 \equiv \frac{x_3}{\mathcal{W}} \qquad \langle v \rangle_{slit} = \frac{H^2 \Delta p}{3\mu L}$$

What about more complicated Newtonian problems?

What does this show?

Tricky step: **Solving for $\underline{v}$ and $\underline{\underline{\tilde{\tau}}}$ can be difficult**

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- non-Newtonian fluids

Solution strategies:

- advanced analytical techniques for solving partial differential equations (PDEs) -see Bird, et a. *Transport Phenomena*, 1960 or 2001, CM5100, MA4515
- numerical techniques (Ansys, *Comsol Multiphysics*)
  - *www.ansys.com*
  - *www.comsol.com/*

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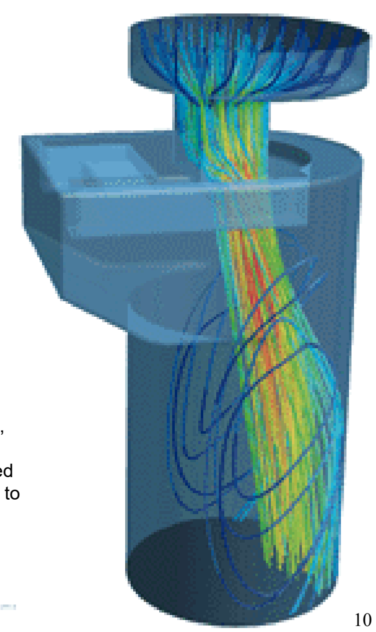
What about more complicated Newtonian problems?

Ansys Fluent : Solutions : Example X19 Fluid Handling and Flow Distribution

Streamlines depict the flow of regenerated catalyst through a slide valve, revealing the source of erosion problems.



"Transport and storage of gases, liquids, or slurries represents a large capital and operating expense in process plants. Fluent's CFD software helps you to design for flow uniformity, balance flows in manifolds, minimize pressure drop, design storage tanks, and accurately size blowers, fans, and pumps. High-speed nozzles and spray systems can be analyzed in order to optimize performance."

www.ansys.com

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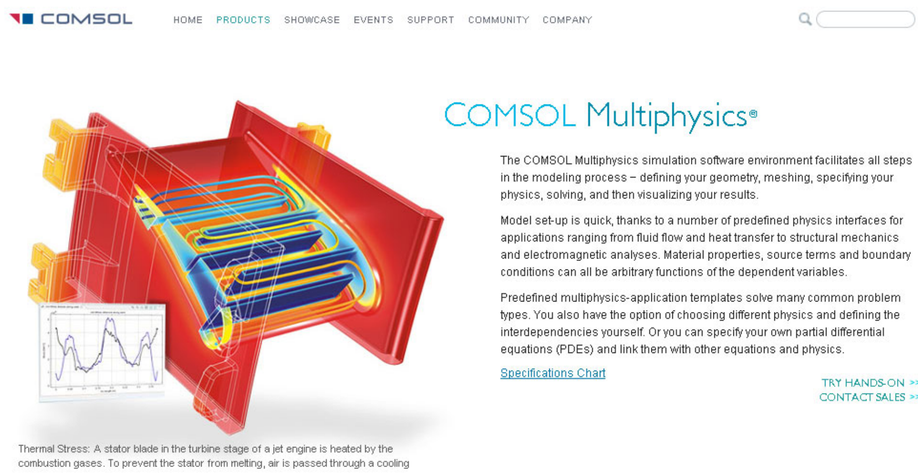
What about more complicated Newtonian problems?

Comsol Multiphysics 5.3



Coming soon: the Comsol Project

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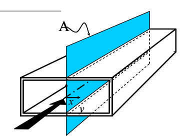
What about more complicated Newtonian problems?

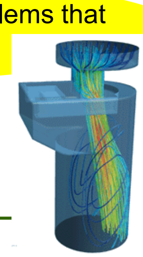


So far:

- We've learned to set-up and sometimes solve flow problems (conservation of mass, momentum)

Question:

- Do we always need to solve the modeling problems that real systems present?
- Can we solve them?



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What about more complicated Newtonian problems?

Most industrial flows are not simple:

- piping
- pumps
- mixers
- flow in an engine
- fluidized beds
- flow in a packed bed (catalytic reactor)
- two-phase flows (extractors)
- jets (jet engines, ink-jet printing)
- coating flows
- evaporators
- heat exchangers

Most of these flows are impossible to solve in detail

Exception: plastics, high viscosity flows

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What about more complicated Newtonian problems?

•Questions:

•Do we always need to solve the modeling problems that real systems present? *-No, not always.*

•Can we solve them? *-No, not always.*

•What do we do instead? *-Experiments, scale-models, and data-correlations*

•What experiments do we do? *•Random experiments and hope for the best*

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What about more complicated Newtonian problems?

•Questions:

•Do we always need to solve the modeling problems that real systems present? *-No, not always.*

•<u>Can</u> we solve them? *-No, not always.*

•What do we do instead? *-Experiments, scale-models, and data-correlations*

•What experiments do we do?

~~*•Random experiments and hope for the best*~~

•Small-scale pilot experiments that can scale to the real system

Better choice, but how?

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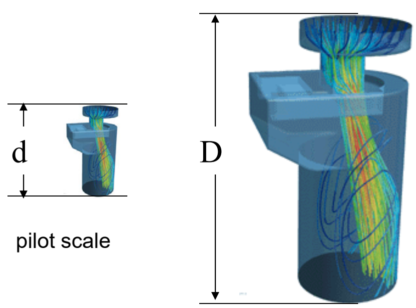
What about more complicated Newtonian problems?

Designing a Device

•Dimensional similarity

= similar proportions



pilot scale

full scale

•Dynamic similarity

= similar behavior

?

How systems **behave** depends on the laws of physics.

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What about more complicated Newtonian problems?

GOALS of Dimensional Analysis:

To use our knowledge of physical laws (mass, momentum, energy conservation) to guide our studies, modeling, and experimentation on **complex (real engineering) flows**
(i.e. to save us trial-and-error work)

Specifically:

- To be able to **design** devices in which the flow is expected to be complex
- To **scale-up** (relate) any experiments to similar flows that are not (yet) available for experimentation
- To guide the use and production of **data correlations** (i.e. the plotting and reporting of experimental data)

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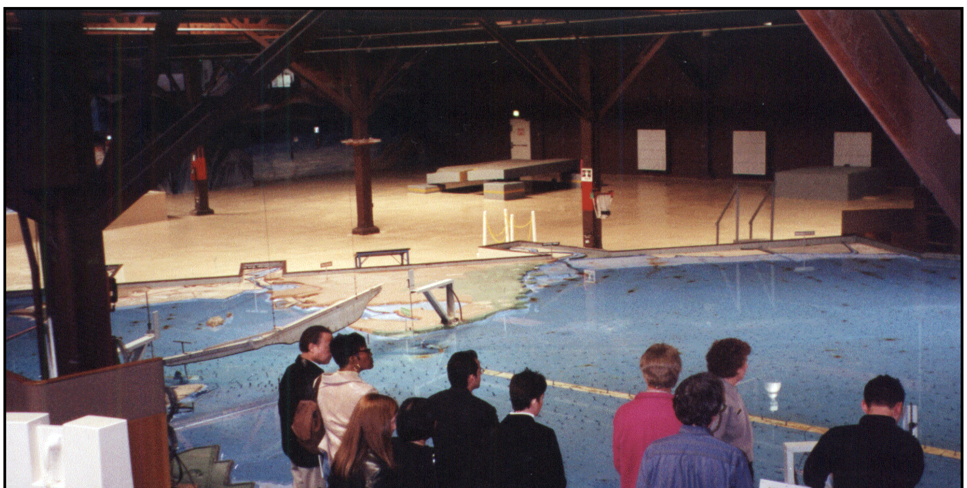

San Francisco Bay Model, Sausalito, CA.

- distorted scale: the dimension in the vertical direction is 1/10th the scale in the horizontal direction.
- US Army Corps of Engineers
- used to evaluate proposed changes to the bay such as dams and other types of development.

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What about more complicated Newtonian problems?

Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

governing equations
- *continuity equation* (mass conservation)
- *equation of motion* (momentum conservation)

Strategy: render the governing equations dimensionless to identify the important parameters that apply in every situation.

$\Longrightarrow$ rely on experiments and data correlations

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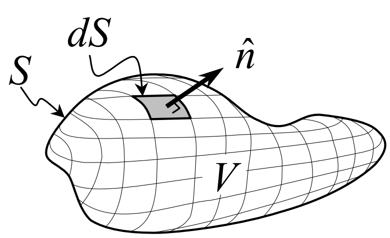
What about more complicated Newtonian problems?

Continuity Equation



Microscopic **mass** balance written on an arbitrarily shaped control volume, *V*, enclosed by a surface, *S*

$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

Gibbs notation: $$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho (\nabla \cdot \underline{v})$$

Microscopic mass balance is a scalar equation.

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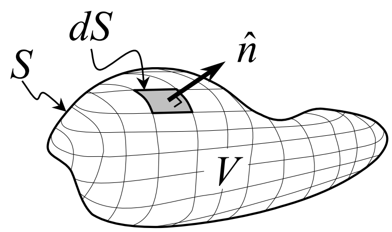
What about more complicated Newtonian problems?

Equation of Motion



Microscopic **momentum** balance written on an arbitrarily shaped control volume, *V*, enclosed by a surface, *S*

Gibbs notation: $$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\nabla\cdot\underline{\underline{\tilde{\tau}}}+\rho\underline{g}$$ **general fluid**

Gibbs notation: $$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$ **Newtonian fluid**

Navier-Stokes Equation

Microscopic momentum balance is a vector equation.

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Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

Mass is conserved:
Continuity Equation

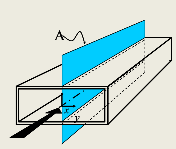
$$\frac{\partial\rho}{\partial t}+\underline{v}\cdot\nabla\rho=-\rho\left(\nabla\cdot\underline{v}\right)$$

Momentum is conserved:
Navier-Stokes Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla P+\mu\nabla^2\underline{v}+\rho\underline{g}$$

For a complex problem,

Which terms dominate?
How can we simplify?



Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

Mass is conserved:
Continuity Equation

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho \left(\nabla \cdot \underline{v} \right)$$

Depends on how big $\underline{v}$ is

Momentum is conserved:
Navier-Stokes Equation

Depends on how big μ is

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla P + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

For a complex problem,

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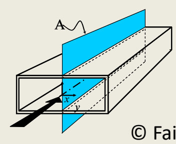

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Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

Mass is conserved:
Continuity Equation

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho \left(\nabla \cdot \underline{v} \right)$$

Depends on how big $\underline{v}$ is

Momentum is conserved:
Navier-Stokes Equation

Depends on how big $\nabla \underline{v}$ is

Depends on how big μ is

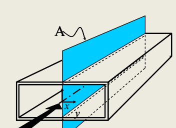
$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla P + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Depends on how fast $\underline{v}$ is changing

Depends on how big $\nabla^2 \underline{v}$ is

For a complex problem,

Which terms dominate?
How can we simplify?



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Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

Mass is conserved:
Continuity Equation

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho (\nabla \cdot \underline{v})$$

Depends on how big $\underline{v}$ is

Momentum is conserved:
Navier-Stokes Equation

Depends on how big $\nabla \underline{v}$ is

Depends on how big μ is

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla P + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

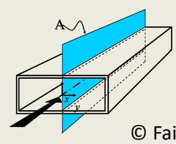
Depends on how big ρ is

Depends on how fast $\underline{v}$ is changing

Depends on how big $\nabla^2 \underline{v}$ is

For a complex problem,

Which terms dominate?
How can we simplify?



Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

For a complex problem,

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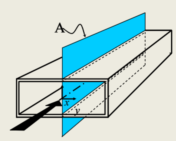

Variables, constants:

$\underline{v}, t, p, x, y, z, \nabla, \nabla^2$

$\mu, \rho, \underline{g}$

- Choose "typical" values (*scale factors*)
- Use them to scale the equations
- Deduce which terms dominate

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Note that once the variables are non-dimensionalized, the scale factors and constants will form **dimensionless groups**

Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

Procedure:

1. select appropriate differential equations and boundary conditions
2. select characteristic quantities with which to scale the variables, e.g. $\underline{v}$, *x*, *P*
   - characteristic quantities must be constant
   - must be representative of the system
3. scale all variables in the governing equations; yields dimensionless equation as a function of ***dimensionless groups***
   The values of the dimensionless groups determine the properties of the differential equations.
4. design scaled-down experiments to develop ***data correlations*** for the system of interest
5. use data correlations to design and evaluate systems

OR

4. perform experiments on an existing system and ***correlate*** results using dimensionless groups

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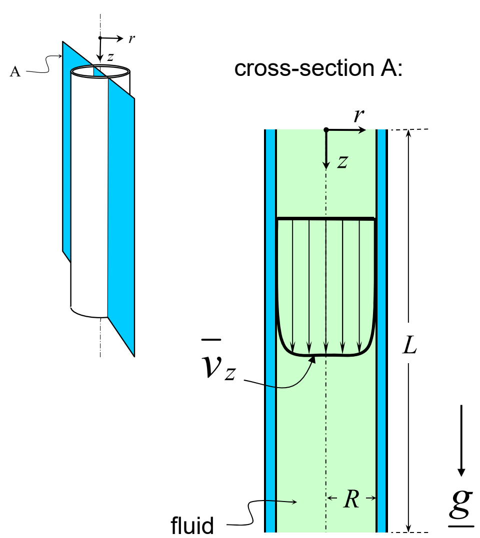
Dimensional Analysis



***EXAMPLE I*:**
Pressure-driven flow of an incompressible Newtonian fluid in a tube:
NOT Laminar (not unidirectional)

- steady state
- well developed
- long tube
- incompressible

locally the flow is 3D:

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z}$$

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Dimensional Analysis

z-component of the Navier-Stokes Equation:

Nothing gets slashed or burned!

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

Need a "Plan B" (Dimensional Analysis)

Choose:

D = characteristic length
V = characteristic velocity
D/V = characteristic time
ρV^2 = characteristic pressure

- Choose "typical" values (*scale factors*)
- Use them to scale the equations
- Deduce which terms dominate

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Dimensional Analysis

non-dimensional variables:

time:	position:	velocity:	driving force:
$t^* \equiv \frac{tV}{D}$	$r^* \equiv \frac{r}{D}$	$v_z^* \equiv \frac{v_z}{V}$	$P^* \equiv \frac{P}{\rho V^2}$
	$z^* \equiv \frac{z}{D}$	$v_r^* \equiv \frac{v_r}{V}$	$g_z^* \equiv \frac{g_z}{g}$
		$v_\theta^* \equiv \frac{v_\theta}{V}$	

- Choose "typical" values (*scale factors*)
- Use them to scale the equations
- **Deduce which terms dominate** ← ***"slash & burn"-like***

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Dimensional Analysis

- Choose "typical" values (*scale factors*)
- Use them to scale the equations
- **Deduce which terms dominate**

z-component of the **nondimensional** Navier-Stokes Equation:

$$\frac{Dv_z^*}{Dt} = -\frac{\partial P^*}{\partial z^*} + \frac{\mu}{\rho V D}\left(\nabla^2 v_z\right)^* + \frac{gD}{V^2} g^*$$

$$\left(\nabla^2 v_z\right)^* \equiv \frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial v_z^*}{\partial r^*}\right) + \frac{1}{r^{*2}}\frac{\partial^2 v_z^*}{\partial \theta^2} + \frac{\partial^2 v_z^*}{\partial z^{*2}}$$

$$\frac{Dv_z^*}{Dt} \equiv \left(\frac{\partial v_z^*}{\partial t^*} + v_r^* \frac{\partial v_z^*}{\partial r^*} + \frac{v_\theta^*}{r^*}\frac{\partial v_z^*}{\partial \theta} + v_z \frac{\partial v_z^*}{\partial z^*}\right)$$

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Dimensional Analysis

Dimensionless Navier-Stokes:

$$\frac{Dv_z^*}{Dt} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\mathrm{Re}}\left(\nabla^{*2} v_z^*\right) + \frac{1}{\mathrm{Fr}} g^*$$

Two dimensionless groups appear:

$$\mathrm{Re} = \frac{\rho V D}{\mu}$$

Reynolds number = ratio of inertial to viscous forces

$$Fr = \frac{V^2}{gD}$$

Froude number = ratio of inertial to gravity forces ("frood")

If for two systems Re and Fr are the same, the two systems are governed by the same momentum, same mass balance.

If the dimensionless boundary conditions are also the same, the two systems are mathematically identical

= Dynamic Similarity

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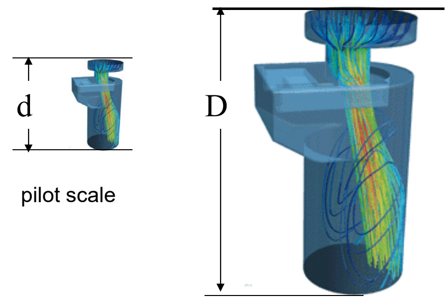
Scale-up

Design & Scale Up

Dimensionless Navier-Stokes:

$$\frac{Dv_z^*}{Dt} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\text{Re}}\left(\nabla^{*2} v_z^*\right) + \frac{1}{\text{Fr}} g^*$$

- **Dimensional** similarity
  - = similar proportions



pilot scale

full scale

- **Dynamic** similarity
  - = similar behavior

We match dimensionless numbers **Re** and **Fr** to achieve this

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Dimensional Analysis

Data Correlations

Dimensionless Navier-Stokes:

$$\frac{Dv_z^*}{Dt} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\text{Re}}\left(\nabla^{*2} v_z^*\right) + \frac{1}{\text{Fr}} g^*$$

We can also use non-dimensionalization to help us to correlate experimental results.

What does it mean to correlate?

34

Dimensionless Navier-Stokes: $\frac{Dv_z^*}{Dt} = -\frac{\partial p^*}{\partial z^*} + \frac{1}{\text{Re}}(\nabla^{*2} v_z^*) + \frac{1}{\text{Fr}} g^*$

If we need to know about the operation of an apparatus *and we have the apparatus*, then we can learn whatever we need to know about the apparatus by conducting experiments.

What if we don't have the apparatus? (we're designing one or comparing the possible performance of one with another)

Answer: we can build a scale model and scale up the experimental results on that;

or

Answer: we can use others' results on scale models and scale up ***their*** experimental results to ***our*** needs (no point in re-inventing the wheel)

Either way, this is called creating a *data correlation*.

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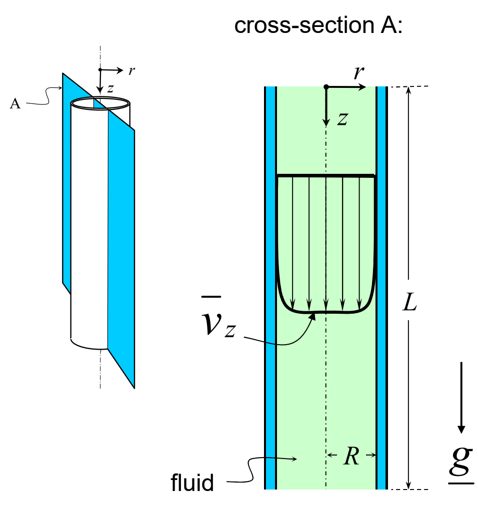
Dimensional Analysis

Creating a Data Correlation:
Choose a model flow



***EXAMPLE*:**
Pressure-driven flow of an incompressible Newtonian fluid in a tube:
NOT Laminar (not unidirectional)

- steady state
- well developed
- long tube
- incompressible

locally the flow is 3D:

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z}$$

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Creating a Data Correlation

To build a correlation, we start with a simple, model flow

Laminar flow in a pipe

For laminar flow of a Newtonian fluid we can calculate the relationship between pressure drop and flow rate. We can also calculate the frictional force on the wall, which is related to these.

$$\underline{F} = \iint_S \left[\hat{n}\cdot\underline{\underline{\Pi}}\right]_{surface} dS$$

(from solution to NS eqns)

$$\tilde{\tau}_{rz} = \frac{r\Delta P}{2L}$$

z-component of force on the wall

$$F_z = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz}\Big|_{r=R} R\, d\theta dz$$

$$= \pi R^2 \Delta P$$

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($v_z(r)$ from solution to NS eqns)

$$Q = \int_0^R \int_0^{2\pi} v_z\, r dr\, d\theta = \pi R^2 \langle v_z \rangle$$

$$= \frac{\pi \Delta P R^4}{8\mu L}$$

Hagen-Poiseuille equation

$$F_z = \Delta P \pi R^2 = \left(\frac{8\mu L}{R^2}\right) Q$$

(Tells us what pump we need for a given flow rate, for example)

For laminar flow in any pipe

What about turbulent flow?

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Creating a Data Correlation

Turbulent flow in a pipe

The frictional force on the wall is again related to this expression:

z-component of force on the wall

$$F_z = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz}\Big|_{r=R} R\,d\theta\,dz$$

but without the solution for $v_z(r)$, where will we get **this**?

If we cannot solve for F_z, how will we get Δp as a function of Q when the flow is turbulent?

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Creating a Data Correlation

We do not know τ_{zr}, but we do know that it comes from the solution to the Navier-Stokes equation and the continuity equation.

(we just cannot solve it because turbulent flow is ***way*** *too complicated.)*

What then?

•We could do experiments.

(but what if we do not have the system? what if it is a design problem?)

•We (or someone else) could do experiments on a similar system and then we could scale the results.

ahhh . . . ***DIMENSIONAL ANALYSIS***

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Creating a Data Correlation

Dimensionless Force on the Wall

$$F_z = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz}\Big|_{r=R} R\, d\theta\, dz$$

$$= \int_0^L \int_0^{2\pi} \mu \left(\frac{\partial v_z}{\partial r} \right)\Bigg|_{r=R} R\, d\theta\, dz$$

Nondimensionalize:

position:

$$r^* \equiv \frac{r}{D}$$

$$z^* \equiv \frac{z}{D}$$

velocity:

$$v_z^* \equiv \frac{v_z}{V}$$

How shall we nondimensionalize F_z?

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Creating a Data Correlation

Nondimensional Wall Friction

$$f \equiv \frac{F_z}{(area)(kinetic\ energy)}$$

$$= \frac{F_z}{(2\pi RL)\left(\frac{1}{2}\rho v^2\right)}$$

Fanning friction factor
dimensionless wall friction in a tube

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Creating a Data Correlation

Non-dimensional force on the wall:

$$f = \frac{1}{\pi}\frac{D}{L}\frac{1}{\text{Re}}\int_0^{\frac{L}{D}}\int_0^{2\pi}\left(\frac{\partial v_z^*}{\partial r^*}\right)\Bigg|_{r^*=\frac{1}{2}} d\theta \, dz^*$$

$$\Rightarrow f = f\left(\text{Re}, \frac{L}{D}\right)$$

for well developed flow expts show there is no L/D dependence

$$\Rightarrow f = f(\text{Re})$$

Conclusion: wall friction, *f,* should only correlate (vary) with ***Re***

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Creating a Data Correlation

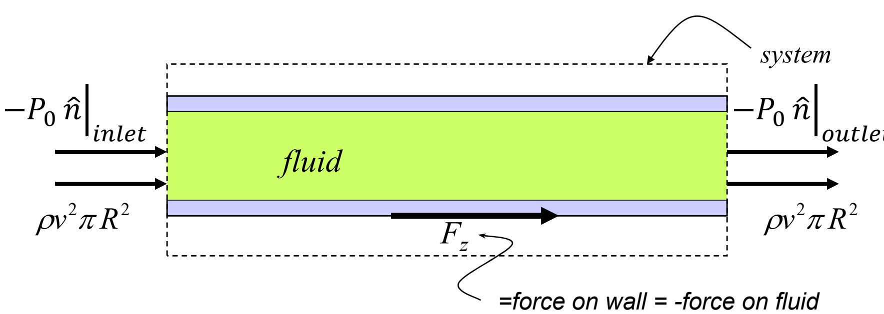
One final question:

How do we measure f ?

How to measure f ?

Answer:

We can see how to measure f by performing a **macroscopic momentum balance** on a straight pipe (incompressible fluid).



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Creating a Correlation

Result of macroscopic momentum balance on straight pipe:

(details in next lectures)

$$F_z = (P_o - P_L)\pi R^2$$

$$f \equiv \frac{F_z}{(area)(kinetic\ energy)} = \frac{F_z}{(2\pi RL)\left(\frac{1}{2}\rho v^2\right)}$$

Fanning Friction Factor

$$f = \frac{(P_o - P_L)\pi R^2}{(2\pi RL)\left(\frac{1}{2}\rho v^2\right)} = \frac{(P_o - P_L)\frac{1}{4}}{\left(\frac{L}{D}\right)\left(\frac{1}{2}\rho v^2\right)}$$

(this is one of the equations in the front cover of the book)

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Data correlation for friction factor (ΔP) versus Re (flow rate) in a pipe

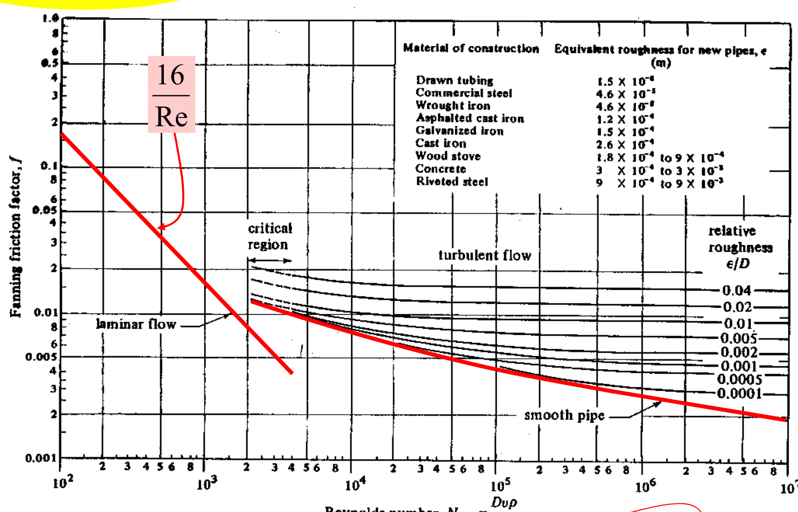
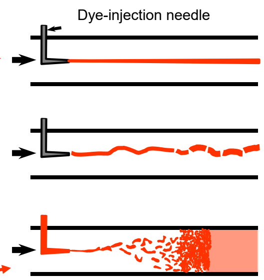


FIGURE 2.10-3. *Friction factors for fluids inside pipes. [Based on L. F. Moody, Trans. A.S.M.E., 66, 671, (1944); Mech. Eng. 69, 1005 (1947). With permission.]*

Moody Chart

(Geankoplis, 2003)

Flow Regimes in a Pipe



Re < 2100 Laminar
- smooth
- one direction only
- predictable

2100< Re < 4000 Transitional

4000 < Re Turbulent
- chaotic - fluctuations within fluid
- transverse motions
- unpredictable - deal with average motion
- most common

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What is the Fanning Friction Factor for Laminar Flow?

$$F_z = \Delta P \pi R^2 = \left(\frac{8\mu L}{R^2}\right) Q$$

$$f = \frac{(P_0 - P_L)\frac{1}{4}}{\left(\frac{L}{D}\right)\left(\frac{1}{2}\rho v^2\right)} = \frac{16\mu}{\rho v D} = \frac{16}{\text{Re}}$$

TRUE!

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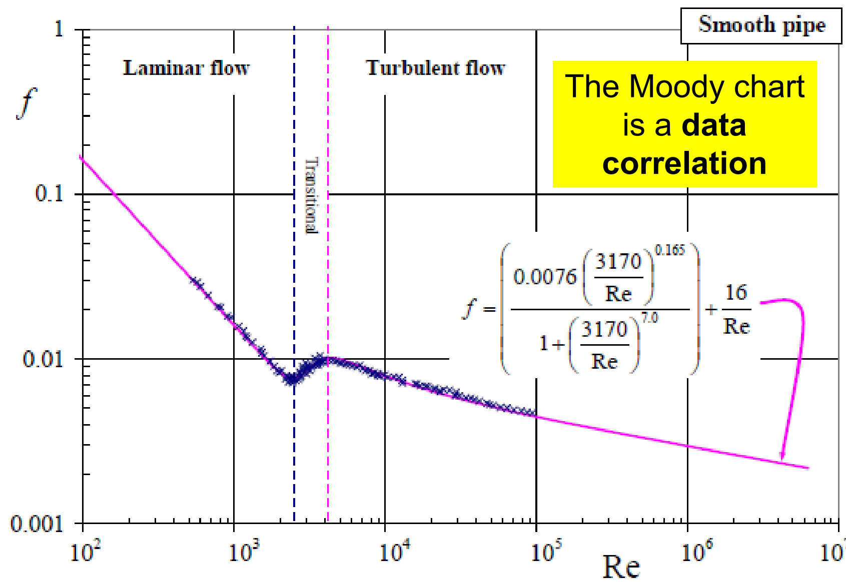

Reference: *Morrison, Faith A, An Introduction to Fluid Mechanics, Cambridge University Press, 2013, Figure 7.18, page 532.*

Q: What have we done so far?

A: learn to non-dimensionalize

WHY?

•When flow problems are too complex for analytical or numerical solution, use experimental data correlations. Non-dimensionalization guides the production and use of these data correlations.

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How can we apply this approach to a new problem?

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How can we apply this approach to a new problem?

Understanding a new system

- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved

Real Flows (*continued*)

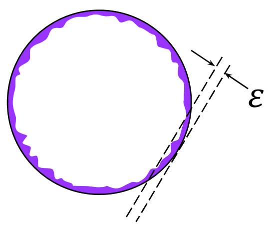
Other internal flows:

•rough pipes - need an additional dimensionless group

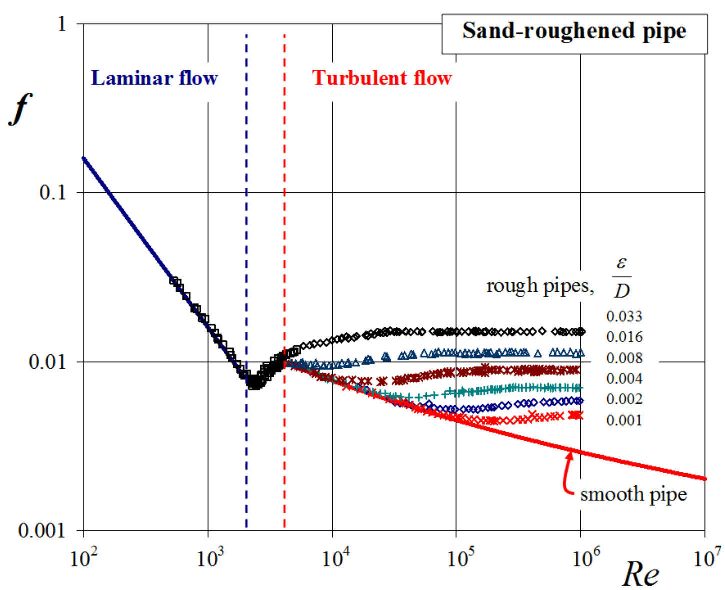
ε- characteristic size of the surface roughness

$\frac{\varepsilon}{D}$ - relative roughness (dimensionless roughness)

$$\frac{1}{\sqrt{f}} = -4.0\log_{10}\frac{\varepsilon}{D} + \frac{4.67}{Re\sqrt{f}} + 2.28$$



Colebrook correlation (Re>4000)



J. Nikuradse, ``Stromungsgesetze in Rauhen Rohren," *VDI Forschungsh*, **361** (1933); English translation, *NACA Tech. Mem. 1292*.

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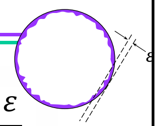
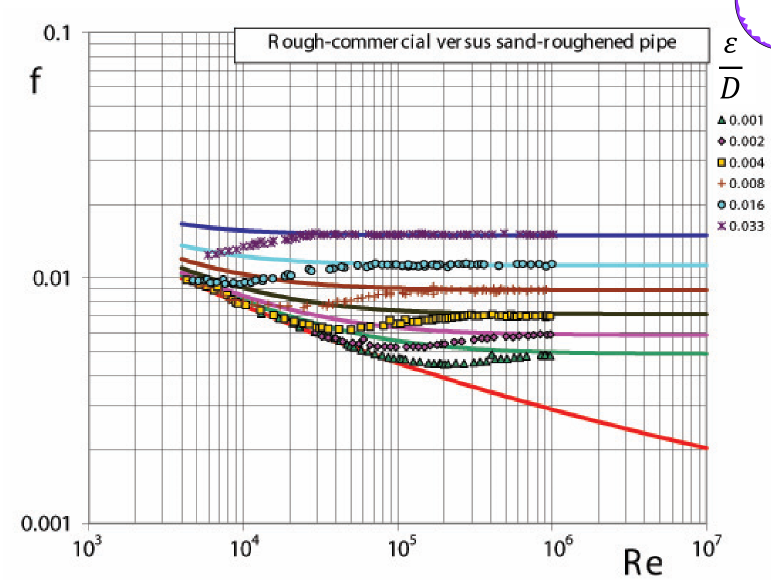
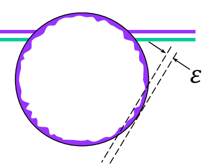
Real Flows – Rough Pipes (*continued*)





J. Nikuradse, ``Stromungsgesetze in Rauhen Rohren," *VDI Forschungsh*, **361** (1933); English translation, *NACA Tech. Mem. 1292*.

Real Flows (*continued*)



Surface Roughness for Various Materials

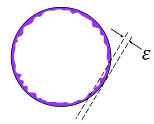
Material	$\varepsilon(mm)$
Drawn tubing (brass,lead, glass, etc.)	1.5x10^{-3}
Commercial steel or wrought iron	0.05
Asphalted cast iron	0.12
Galvanized iron	0.15
Cast iron	0.46
Wood stave	0.2-.9
Concrete	0.3-3
Riveted steel	0.9-9

from Denn, Process Fluid Mechanics, Prentice-Hall 1980; p46

Real Flows (*continued*)

Other internal flows:

•rough pipes -

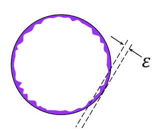


What else?

Real Flows (*continued*)

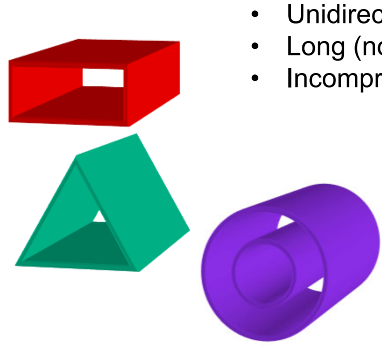
Other internal flows:

•rough pipes -



•flow through noncircular conduits

- Steady
- Unidirectional
- Long (no end effects)
- Incompressible



Let's go back through the analysis and see where we assumed the pipe was circular

Straight Pipe:

Result of macroscopic momentum balance on straight pipe:

$$F_z = (P_o - P_L)\pi R^2$$

$$f \equiv \frac{F_z}{(area)(kinetic\ energy)} = \frac{F_z}{(2\pi RL)\left(\frac{1}{2}\rho V^2\right)}$$

Fanning Friction Factor

$$f = \frac{(P_o - P_L)\pi R^2}{(2\pi RL)\left(\frac{1}{2}\rho v^2\right)} = \frac{(P_o - P_L)\frac{1}{4}}{\left(\frac{L}{D}\right)\left(\frac{1}{2}\rho v^2\right)}$$

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Straight Pipe: where did we assume circular?

Result of macroscopic momentum balance on straight pipe:

HERE →

$$F_z = (P_o - P_L)\pi R^2$$

$$f \equiv \frac{F_z}{(area)(kinetic\ energy)} = \frac{F_z}{(2\pi RL)\left(\frac{1}{2}\rho V^2\right)}$$

and HERE → $(2\pi RL)$

If we change these to general relations (good for any shape, . . .

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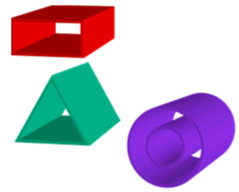
Real Flows (*continued*)

Other internal flows:

•flow through noncircular conduits



We can show:

Drag in conduit:

$$F_{drag} = F_z = \Delta P A_{crosssection}$$

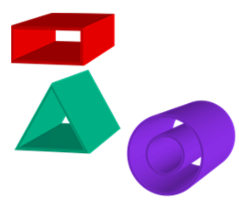
Wetted surface:

$$(perimeter)L$$

Real Flows (*continued*)

Other internal flows:

•flow through noncircular conduits



We can show:

Drag in conduit:

$$F_{drag} = F_z = \Delta P A_{crosssection}$$

Wetted surface:

$$(perimeter)L$$

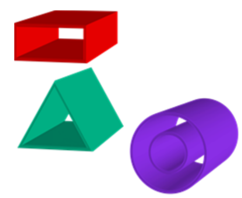
Carrying out the dimensional analysis, we see that a good characteristic length is given by:

$$D \equiv D_H = \frac{4A_{xs}}{perimeter}$$

Real Flows (*continued*)

Other internal flows:

•flow through noncircular conduits



We can show:

Drag in conduit:

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Wetted surface:

$$(perimeter)L$$

Carrying out the dimensional analysis, we see that a good characteristic length is given by:

$$D \equiv D_H = \frac{4A_{xs}}{perimeter}$$

It works! (for both laminar and turbulent)

Real Flows (*continued*)

Other internal flows:

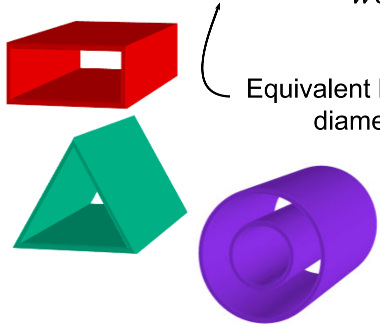
•flow through noncircular conduits

Empirically, it is found that f vs. Re correlations for circular conduits matches the data for noncircular conduits if D is replaced with equivalent hydraulic diameter D_H.

$$D_H \equiv (\frac{4(crosssectional\ area)}{wetted\ perimeter} = 4R_H$$

Equivalent hydraulic diameter

Hydraulic radius

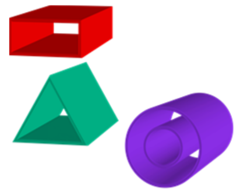


Real Flows (*continued*) •flow through noncircular conduits

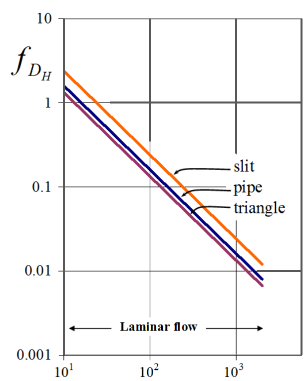
Laminar flow

$$f_{D_H} Re_{D_H} = constant = Po$$

Po ≡Poiseuille number



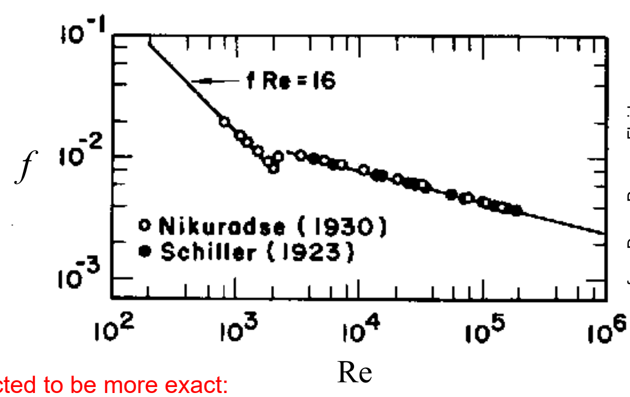
$\underline{Po}$
Circle=16
Slit=24
Ellipse=function of a, b
Triangle=13.33



Real Flows (*continued*)

Flow Through Noncircular Conduits - Turbulent

•Flow through equilateral triangular conduit
•f and Re calculated with D_H
•solid lines are for circular pipes

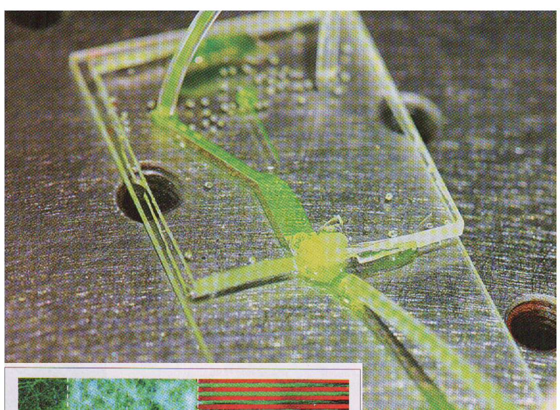


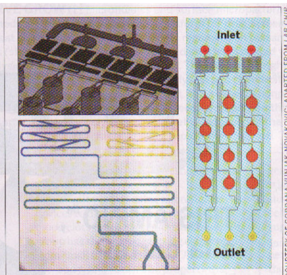
from Denn, Process Fluid Mechanics, Prentice-Hall 1980; p48

Can be corrected to be more exact:
See section 7.2.2 (Morrison, p570)

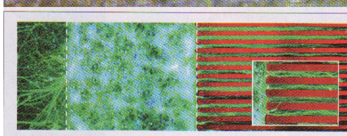
$$\frac{1}{\sqrt{f_{D_H}}} = 4.0 \log \frac{\left(Re_{D_H}\sqrt{f_{D_H}}\right)}{\frac{Po_{duct}}{16}} - 0.40$$

Non-Circular Cross-sections have application in the new field of microfluidics





BEST OF BOTH The microbioreactor array combines the independence of the petri dish with the flow and reagent control of microfluidic devices. Three views of the device are shown: a schematic of the entire device (right), an artist's rendition of the inlet end (top left), and a micrograph of the region where two channels meet.



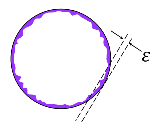
NERVE GROWTH Microfluidics and surface micropatterning methods are combined to develop a platform for drug screening to treat spinal cord injury and for axonal regeneration research. Cells (stained blue) are selectively placed on a band coated with polylysine (between dashed white lines). The area to the right is patterned with alternating strips of polylysine and chondroitin sulfate proteoglycan. The area to the left is coated with nonpatterned polylysine. The axons (stained green) grow only on the sections coated with polylysine. The inset is a closeup of the junction between the patterned and nonpatterned regions.

Chemical & Engineering News, 10 Sept 2007, p14

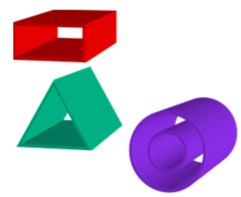
Real Flows (*continued*)

Other internal flows:

- rough pipes -



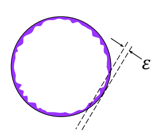
- flow through noncircular conduits



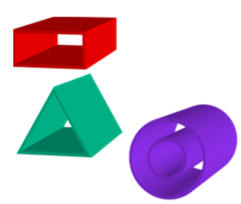
Real Flows (*continued*)

Other internal flows:

- rough pipes -



- flow through noncircular conduits



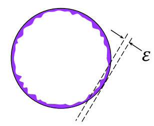
What else?

© Faith A. Morrison, Michigan Tech U.

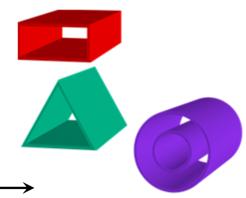
Real Flows (*continued*)

Other internal flows:

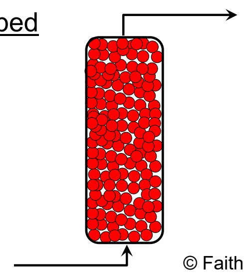
•rough pipes -



•flow through noncircular conduits



•flow through a packed bed



Commonly used as

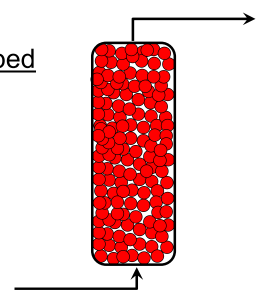
- Reactors
- Separators

(distillation columns, absorbers)

Real Flows (*continued*)

Other internal flows:

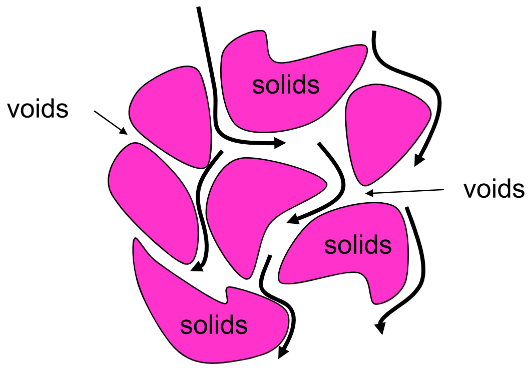
•flow through a packed bed

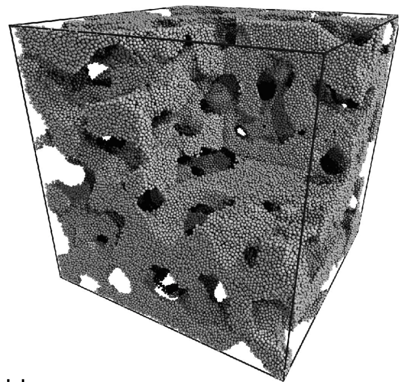


Understanding a new system

- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved

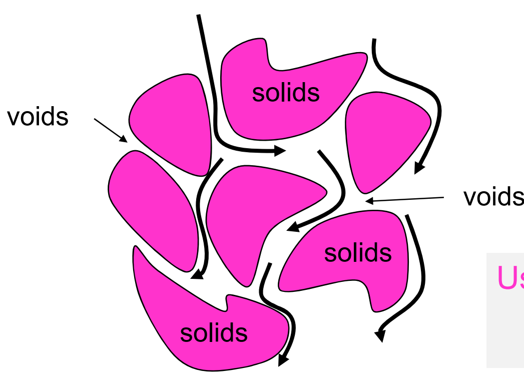
Flow Through Packed Beds

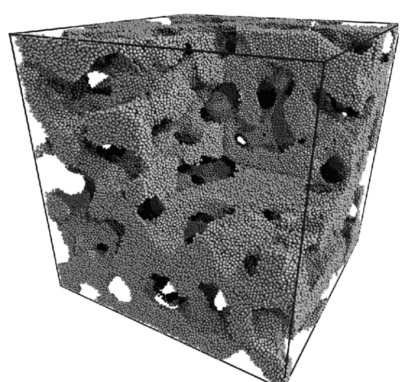




Flow Through Packed Beds

Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit





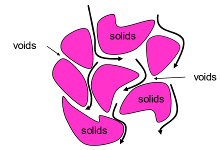
Use Hydraulic Diameter

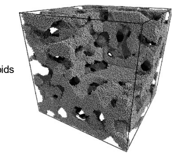
$$D_H \equiv \frac{4A_{xs}}{p}$$

(see Example 7.16, page 564)

Flow Through Packed Beds

Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit





$$D_H \equiv \frac{4A_{xs}}{\not{p}} = \frac{4\varepsilon}{(1-\varepsilon)a_v}$$

(see text for details)

ε = void fraction
a_v = specific surface area of the packing
v_0 = superficial velocity= $\frac{Q}{Vol/L}$

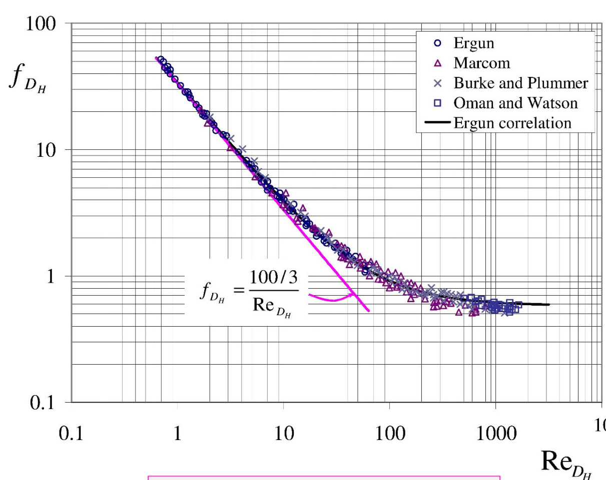
$$f_{D_H} = \left(\frac{\Delta p}{L}\right)\frac{D_H \varepsilon^2}{2\rho v_0^2}$$

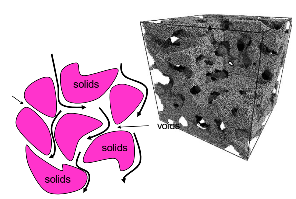
$$\mathrm{Re}_{D_H} = \frac{\rho V D_H}{\mu}$$

Does it work?

(see Example 7.16, page 564)

Flow Through Packed Beds

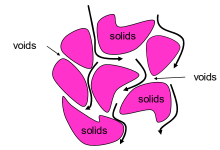


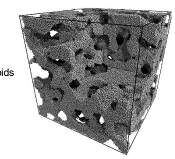


$$\frac{100/3}{\mathrm{Re}_{D_H}} + \frac{1.75}{3} = f_{D_H}$$

Ergun Equation
Friction factor/Re number relationship for flow through packed beds

Flow Through Packed Beds





Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit

Ergun Equation
Friction factor/Re number relationship for flow through packed beds

$$\frac{100/3}{\mathrm{Re}_{D_H}} + \frac{1.75}{3} = f_{D_H}$$

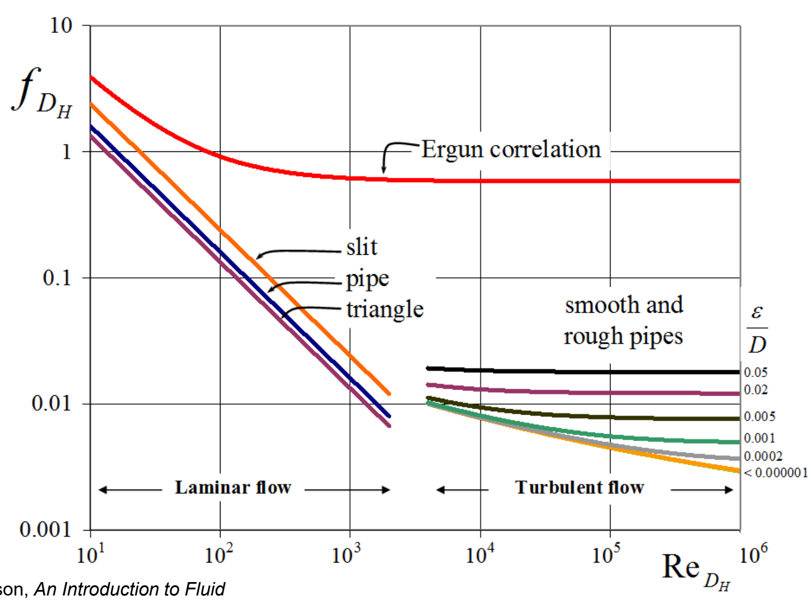
(see text for details)

$$f_{D_H} = \left(\frac{\Delta p}{L}\right)\frac{D_H \varepsilon^2}{2\rho v_0^2}$$

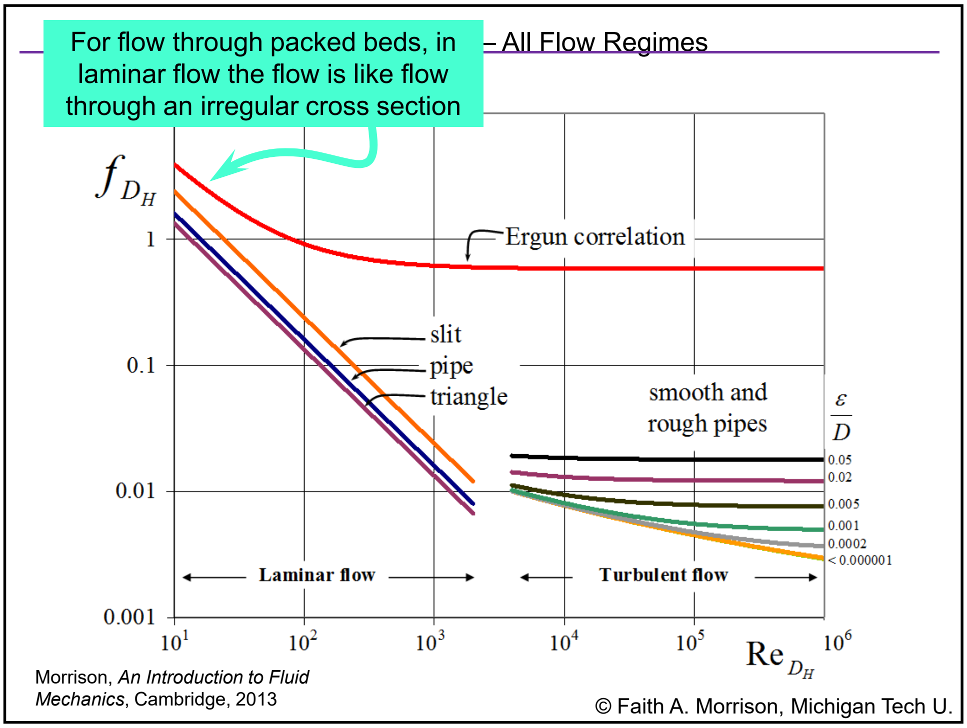
$$\mathrm{Re}_{D_H} = \frac{\rho V D_H}{\mu}$$

(see Example 7.16, page 564)

Flow Through Noncircular Conduits – All Flow Regimes



Morrison, *An Introduction to Fluid Mechanics*, Cambridge, 2013


For flow through packed beds, in laminar flow the flow is like flow through an irregular cross section
– All Flow Regimes
f_{D_H}
Ergun correlation
slit
pipe
triangle
smooth and rough pipes
$\frac{\varepsilon}{D}$
0.05
0.02
0.005
0.001
0.0002
< 0.000001
1
0.1
0.01
0.001
Laminar flow
Turbulent flow
10^1
10^2
10^3
10^4
10^5
10^6
Re_{D_H}
Morrison, *An Introduction to Fluid Mechanics*, Cambridge, 2013
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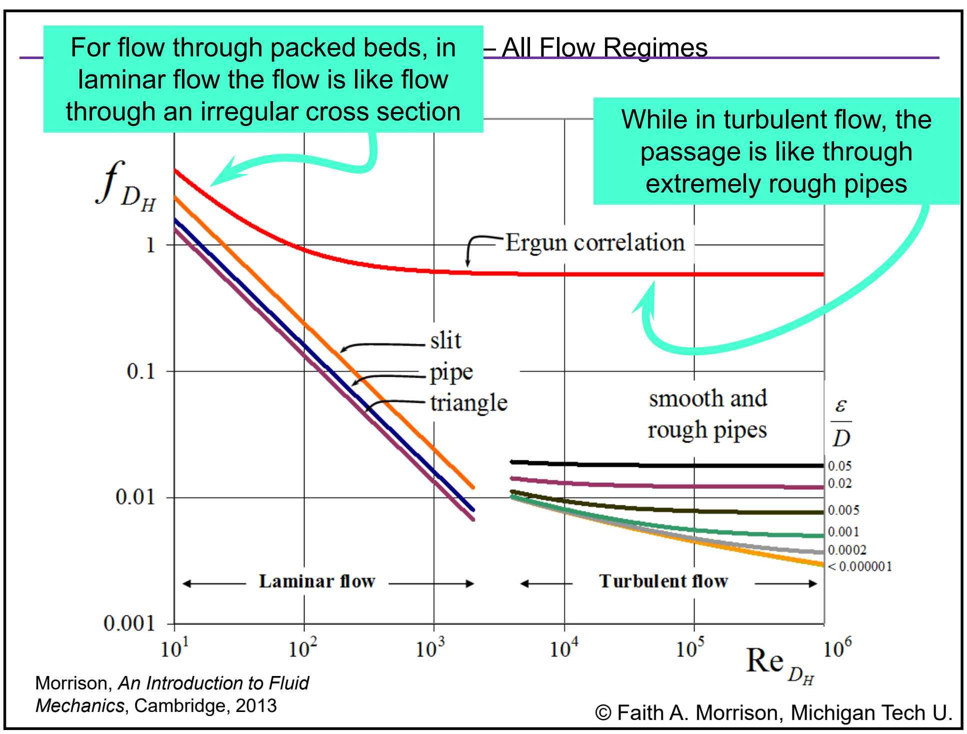
For flow through packed beds, in laminar flow the flow is like flow through an irregular cross section
– All Flow Regimes
While in turbulent flow, the passage is like through extremely rough pipes
f_{D_H}
Ergun correlation
slit
pipe
triangle
smooth and rough pipes
$\frac{\varepsilon}{D}$
0.05
0.02
0.005
0.001
0.0002
< 0.000001
1
0.1
0.01
0.001
Laminar flow
Turbulent flow
10^1
10^2
10^3
10^4
10^5
10^6
Re_{D_H}
Morrison, *An Introduction to Fluid Mechanics*, Cambridge, 2013
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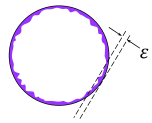
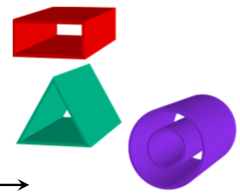
Summary

Other internal flows:

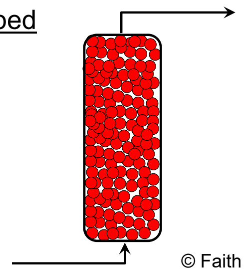
•rough pipes -



•flow through noncircular conduits



•flow through a packed bed



Commonly used as
- Reactors
- Separators

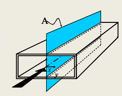
(distillation columns, absorbers)

Summary

- Complex flows are governed by the same *physics* as simple flows, but the math (programming) is harder
- We can leverage our knowledge about the *physics* through **Dimensional Analysis (DA)**

For a complex problem,

Which terms dominate?
How can we simplify?



- **DA** reveals the dimensionless numbers that govern dynamic similarity, e.g. Re, Fr
- Experiments on geometrically and dynamically similar systems can be used to correlate results, e.g. Moody plot, and to perform scale-up, e.g. pilot plant studies.

Summary *(continued)*

- In addition to using the data correlations of others, we can use **Dimensional Analysis** to analyze new, never-before-studied systems (like in a plant, or novel device design)

Understanding a new system

- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved

We discussed how this procedure worked for: (1) rough pipes, (2) noncircular cross sections, (3) flow through packed beds (all internal flows, Chapter 7).

We will also use this procedure for external flows: skydiving, automotive drag, etc. (Chapter 8)

Next: