Before continuing on to External flows

Let's take the time to explore a different type of momentum balance

CM3110
Transport I
Part I: Fluid Mechanics

Michigan Tech

More Complicated Flows II: External Flow
(or applying fluid-mechanics problem-solving to a new category of flows)

Professor Faith Morrison

Department of Chemical Engineering
Michigan Technological University

CM3110
Transport I
Part I: Fluid Mechanics

Michigan Tech

Macroscopic Momentum Balances

Professor Faith Morrison

Department of Chemical Engineering
Michigan Technological University

Michigan Tech

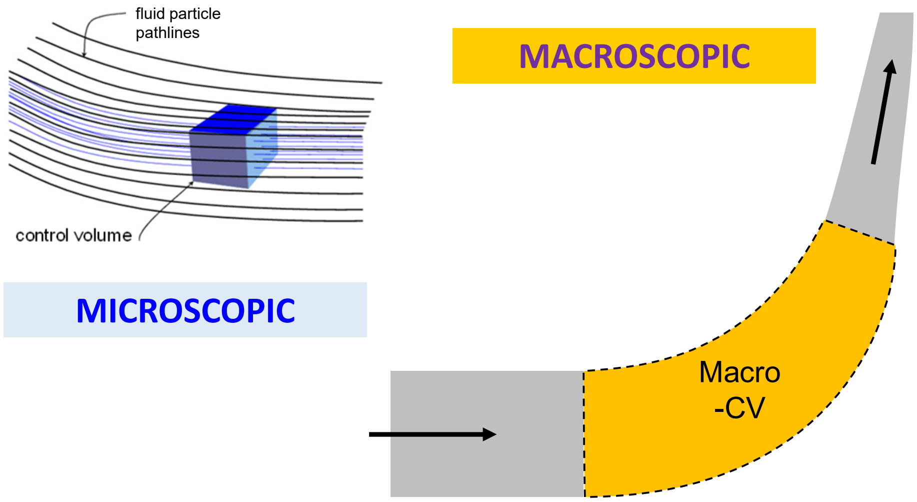
CM3110
Transport I
Part I: Fluid Mechanics

Macroscopic Momentum Balances

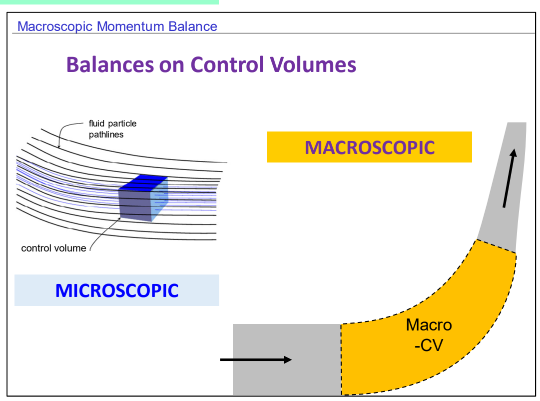
Professor Faith Morrison

Department of Chemical Engineering
Michigan Technological University

Balances on Control Volumes



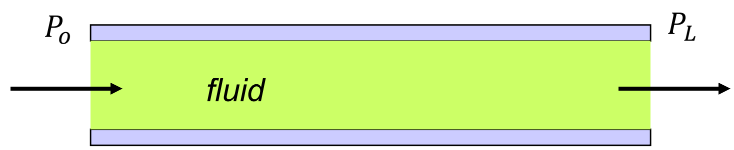
What kinds of problems are best approached with the macroscopic momentum balance?



Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Drag on the walls of a pipe

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

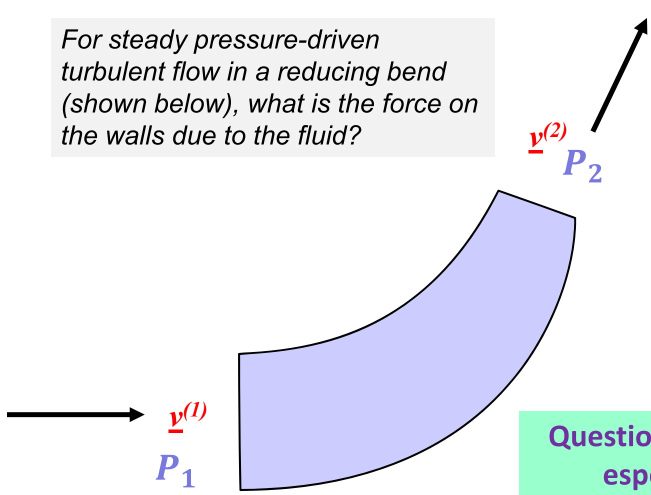


Questions about devices, especially *forces*.

Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?



Questions about devices, especially *forces*.

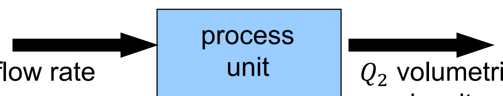
Macroscopic Momentum Balance

Macroscopic Balances

- Use when we do not need the details of the velocity profile
- 3 types:
  - -mass (CM2110, Felder and Rousseau)
  - -momentum ←Now
  - -energy (CM2110, Felder and Rousseau)

Macroscopic Mass Balance:

Q_1 volumetric flow rate
ρ_1, density



Q_2 volumetric flow rate
ρ_2, density

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$\underline{\mathbf{P}}$ = fluid momentum vector

$\beta_{laminar} = 0.75$
$\beta_{turbulent} \sim 1$

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos\theta \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n} \right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$\underline{R}$ = net force on fluid due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal

See inside front Cover of Morrison, 2013

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$\underline{\mathbf{P}}$ = *fluid* momentum

$$\beta_{laminar} = 0.75$$
$$\beta_{turbulent} \sim 1$$

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos\theta \langle v\rangle^2}{\beta}\hat{\underline{v}}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{\underline{n}}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

(we almost always use it in cartesian coordinates)

$$\begin{pmatrix} \frac{d\mathbf{P}_x}{dt} \\ \frac{d\mathbf{P}_y}{dt} \\ \frac{d\mathbf{P}_z}{dt} \end{pmatrix}_{xyz} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos\theta \langle v\rangle^2}{\beta}\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix}_{xyz}\right]_{A_i}$$

$$= \sum_{i=1}^{\#streams} \left[-pA\begin{pmatrix} \hat{n}_x \\ \hat{n}_y \\ \hat{n}_z \end{pmatrix}_{xyz}\right]_{A_i} + \begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix}_{xyz} + M_{CV}\begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}_{xyz}$$

$\underline{R}$ = net force *on fluid* due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal

21Oct2015

See inside front cover of Morrison, 2013

https://pages.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Compare with the (more familiar) Navier-Stokes

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A \cos\theta \langle v\rangle^2}{\beta}\hat{\underline{v}}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{\underline{n}}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Macroscopic Momentum Balance

Convective terms

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma **Convective terms** ntum **Pressure forces**

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v}\right) = -\nabla \mathrm{p} + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma **Convective terms** ntum **Pressure forces** **Viscous forces**

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v}\right) = -\nabla \mathrm{p} + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma ntum

Convective terms

Pressure forces

Viscous forces

Gravity force

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

Macroscopic Momentum Balance

Macroscopic Momentum Balance

Now we need to learn when and how to apply this

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

Macroscopic Mass Balance

GOAL: Learn to use the macroscopic momentum balance

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Three questions:

1. **Why does it work?**
2. **When do we use it?**
3. **How do we use it?**

We begin with the

Macroscopic Mass Balance

Mass accumulation = Mass in - Mass out

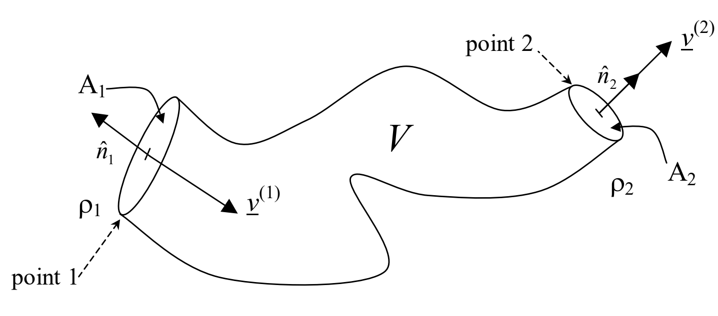
To obtain a general equation (first for mass, then for momentum) we first consider the following case:

- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV

Macroscopic Mass Balance:

Mass in = Mass out

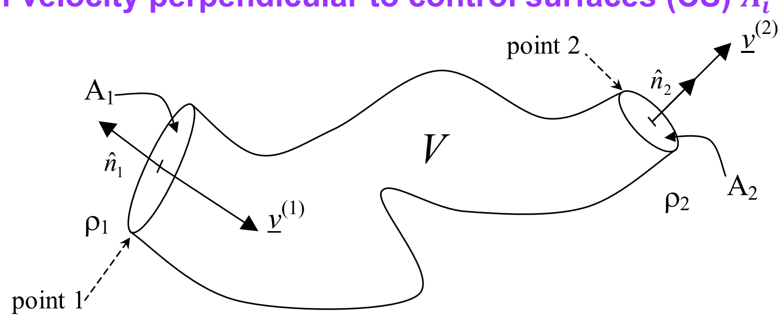
- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV



Macroscopic Mass Balance:

See Chapter 9 for detailed derivation

Arbitrary, single-input, single-output system: *special case* of velocity perpendicular to control surfaces (CS) A_i



Special case:

Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **perpendicular to** A_i
- ρ constant across surface

Macroscopic Mass Balance:

Mass in = Mass out

$$\rho_1 \langle v^{(1)} \rangle A_1 = \rho_2 \langle v^{(2)} \rangle A_2$$

average velocity through surface A_1

cross-sectional area, in

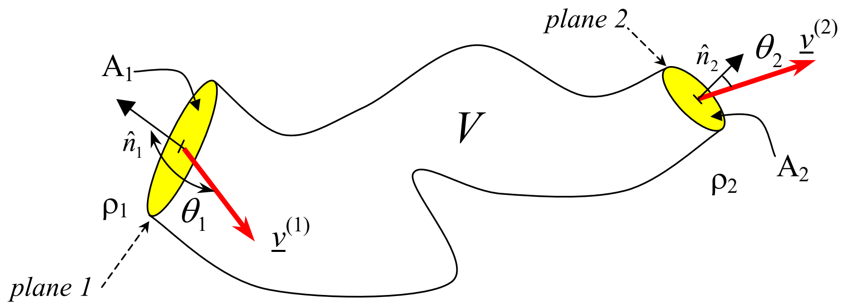
average velocity through surface A_2

cross-sectional area, out

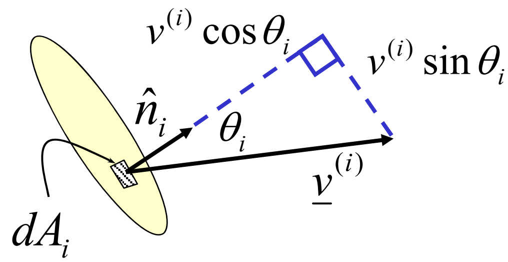
Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **perpendicular to A_i**
- ρ constant across surface

Arbitrary, single-input, single-output system:

velocity is *NOT perpendicular* to control surfaces A_i



Macroscopic Mass Balance



$\hat{\boldsymbol{n}}_{\boldsymbol{i}}$ = ***outwardly*** **pointing (with respect to the control volume CV) unit normal**

Macroscopic Mass Balance

Macroscopic Mass Balance:

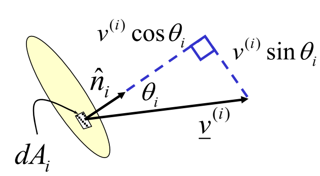
This takes care of 'out' or 'in'

0= net mass out

$$0 = \rho_1 \langle v^{(1)} \rangle \cos\theta_1 A_1 + \rho_2 \langle v^{(2)} \rangle \overbrace{\cos\theta_2}\, A_2$$

Assumptions:

- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **NOT perpendicular to A_i**
- ρ_i constant across surface

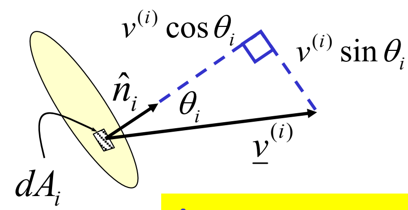


$\hat{n}_i$ = *outwardly* pointing unit normal

Macroscopic Mass Balance

Reminder:

θ relates to the orientation of inlet and outlet surfaces in the chosen coordinate system



$\hat{n}_i =$ *outwardly* pointing unit normal at in/outlet of CV

Macroscopic Momentum Balance

Macroscopic Momentum Balance:

steady state

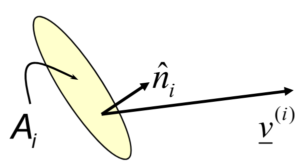
$$\sum F_{on\,CV} = \cancel{\frac{d\mathbf{P}}{dt}} + \underbrace{\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS}_{\left(\substack{\text{net momentum}\\ \text{convected out}}\right)}$$

Momentum balance on fluid in a control volume

We can specialized the ***convective term*** for macroscopic control volumes

$$\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{i=1}^{N}\left[\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i$$

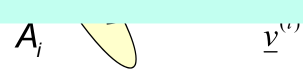
N bounding control surfaces



Macroscopic Momentum Balance

steady state

See Chapter 9 for detailed derivation



N bounding control surfaces

Macroscopic Momentum Balance

Macroscopic Momentum Balance
(Convective term, specialized for macro CV balances)

$$\begin{pmatrix} \text{net momentum} \\ \text{out of CV} \end{pmatrix} = \iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i$$

Input, output surfaces A_i

$$\sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i = \sum_i \iint_{A_i}\left(\rho_i\underline{v}^{(i)}\right)\left(\hat{n}_i\cdot\underline{v}^{(i)}\,dA\right)$$

We can now specify for each A_i:

$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

We now separate velocity *magnitude* from the direction

Macroscopic Momentum Balance-Convective Term

We now separate velocity *magnitude* from the direction

For each inlet or outlet surface A_i:

$$\boxed{\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}}$$

- $v^{(i)}$ = magnitude of velocity through A_i
- $\hat{v}^{(i)}$ = unit vector in direction of velocity through A_i
- $\hat{n}\cdot\underline{v}^{(i)} = v^{(i)}\cos(\theta_i)$ = component of velocity "through" A_i

Macroscopic Momentum Balance-Convective Term

$$\begin{pmatrix} \textit{net momentum} \\ \textit{out of CV} \end{pmatrix} = \iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i$$

Input, output surfaces A_i

$$\sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i = \sum_i \iint_{A_i}\left(\rho\underline{v}^{(i)}\right)\left(\hat{n}_i\cdot\underline{v}^{(i)}dA\right)$$

$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

$$\hat{n}\cdot\underline{v}^{(i)} = v^{(i)}\cos\theta_i$$

We now separate velocity *magnitude* from the direction

Macroscopic Momentum Balance-Convective Term

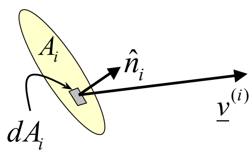
group the velocity magnitudes together

$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

$$\begin{pmatrix} \text{net momentum} \\ \text{convected out} \end{pmatrix} = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)}\right)\left(\underbrace{\hat{n}_i \cdot \underline{v}^{(i)}}_{v^{(i)}\cos\theta_i} dA\right)$$

$$= \sum_i \iint_{A_i} \left(\rho_i v^{(i)}\hat{v}^{(i)}\right)\left(v^{(i)}\cos\theta_i \, dA_i\right)$$

$$= \sum_i \rho_i \hat{v}^{(i)} \cos\theta_i \left(\iint_{A_i} \left(v^{(i)}\right)^2 dA_i \right)$$



only the velocity magnitudes vary across dA_i; they appear as ${v^{(i)}}^2$

Macroscopic Momentum Balance-Convective Term

group the velocity magnitudes together

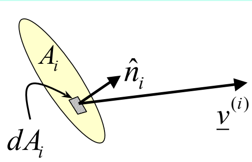
$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

$$\begin{pmatrix} \text{net momentum} \\ \text{convected out} \end{pmatrix} = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)}\right)\left(\underbrace{\hat{n}_i \cdot \underline{v}^{(i)}}_{v^{(i)}\cos\theta_i} dA\right)$$

$$= \sum_i \iint_{A_i} \left(\rho_i v^{(i)}\hat{v}^{(i)}\right)\left(v^{(i)}\cos\theta_i \, dA_i\right)$$

We have assumed that the direction of $\underline{v}^{(i)}$ does not vary across A_i.

$$= \sum_i \rho_i \hat{v}^{(i)} \cos\theta_i \left(\iint_{A_i} \left(v^{(i)}\right)^2 dA_i \right)$$



only the velocity magnitudes vary across dA_i; they appear as ${v^{(i)}}^2$

Macroscopic Momentum Balance

Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **NOT perpendicular to A_i**
- ρ_i constant across surfaces
- $\hat{v}^i$ constant across surfaces

Sum of the forces on the fluid in the CV

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \left[\iint_{A_1} \left(v^{(1)}\right)^2 dA \right] - \rho_2 \cos\theta_2 \hat{v}^{(2)} \left[\iint_{A_2} \left(v^{(2)}\right)^2 dA \right] + \sum_i \underline{F}_{i,on}$$

Macroscopic Momentum Balance

We can write these terms compactly as $\frac{\langle v\rangle^2}{\beta}$, as we now show

Sum of the forces on the fluid in the CV

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \left[\iint_{A_1} \left(v^{(1)}\right)^2 dA \right] - \rho_2 \cos\theta_2 \hat{v}^{(2)} \left[\iint_{A_2} \left(v^{(2)}\right)^2 dA \right] + \sum_i \underline{F}_{i,on}$$

Recall that the average of a function f across a surface A is calculated from:

$$\langle f(x,y)\rangle = \frac{\iint_A f\, dA}{\iint_A dA} = \frac{1}{A}\iint_A f\, dA$$

Macroscopic Momentum Balance

Sum of the forces on the fluid in the CV

$$0=-\rho_1\cos\theta_1\hat{v}^{(1)}\underbrace{\left[\iint_{A_1}\left(v^{(1)}\right)^2 dA\right]}_{=\left\langle\left(v^{(1)}\right)^2\right\rangle A_1}-\rho_2\cos\theta_2\hat{v}^{(2)}\underbrace{\left[\iint_{A_2}\left(v^{(2)}\right)^2 dA\right]}_{=\left\langle\left(v^{(2)}\right)^2\right\rangle A_2}+\sum_i \underline{F}_{i,on}$$

$$\boxed{0=-\rho_1 A_1\left\langle\left(v^{(1)}\right)^2\right\rangle\cos\theta_1\hat{v}^{(1)}-\rho_2 A_2\left\langle\left(v^{(2)}\right)^2\right\rangle\cos\theta_2\hat{v}^{(2)}+\sum_i \underline{F}_{i,on}}$$

But what is this?

We can make this look more like other convective terms we have seen by introducing a factor relating $\langle v^2\rangle$ to average velocity squared.

Macroscopic Momentum Balance

β quantifies the variation of the true velocity profile from **plug flow** (flat profile). $\beta\equiv$ ***Velocity Correction Factor***

$$\sum_i \underline{F}_{i,on}=\rho_1 A_1\left\langle\left(v^{(1)}\right)^2\right\rangle\cos\theta_1\hat{v}^{(1)}+\rho_2 A_2\left\langle\left(v^{(2)}\right)^2\right\rangle\cos\theta_2\hat{v}^{(2)}$$

define: $\boxed{\beta\equiv\frac{\langle v\rangle^2}{\langle v^2\rangle}}$

experimental result

$\beta_{turbulent}=0.95\text{-}0.99$

$\beta_{laminar}=0.75$

Result: Steady State Macroscopic Momentum Balance (convective terms)

$$\boxed{\sum_i \underline{F}_{i,on}=\frac{\rho_1 A_1\left\langle v^{(1)}\right\rangle^2\cos\theta_1}{\beta_1}\hat{v}^{(1)}+\frac{\rho_2 A_2\left\langle v^{(2)}\right\rangle^2\cos\theta_2}{\beta_2}\hat{v}^{(2)}}$$

***vector** equation*

Macroscopic Momentum Balance-Force Terms

Force Terms

$$\boxed{\sum_i \underline{F}_{i,on} = \frac{\rho_1 A_1 \left\langle v^{(1)} \right\rangle^2 \cos\theta_1}{\beta_1} \hat{v}^{(1)} + \frac{\rho_2 A_2 \left\langle v^{(2)} \right\rangle^2 \cos\theta_2}{\beta_2} \hat{v}^{(2)}}$$

Sum of the forces on the fluid in the CV

$$\sum_i \underline{F}_{i,on} = contact + noncontact$$

$$\underline{F}_{contact} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

$$noncontact = M_{CV}\,\underline{g}$$

gravity

Molecular forces

(viscosity and pressure)

Macroscopic Momentum Balance-Force Terms

Contact Forces = pressure + viscous

Viscous: $\underline{R}$ — This is the force on the fluid (force on walls is $-\underline{R}$)

Pressure:

$$\underline{F}_{contact} = \iint_S \left[\hat{n} \cdot \underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

$$= \sum_i \left[\iint_S \left[\hat{n} \cdot (-p\underline{\underline{I}})\right] dS \right]_i$$

$$= \sum_i \left[(-p)\hat{n}\right] \left[\iint_A dS \right]_i$$

$$= \sum_i \left[(-p)\hat{n}A\right]$$

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$\underline{\underline{P}}$ = fluid momentum

$\beta_{laminar} = 0.75$
$\beta_{turbulent} \sim 1$

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$\underline{R}$ = net force on fluid due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal of the macroscopic control volume, CV

See inside front Cover of Morrison, 2013
And the exam formula sheet

Compare with the (more familiar) Navier-Stokes

Macroscopic Momentum Balance

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \mathrm{p} + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt}+\sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i}=\sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i}+\underline{R}+M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial\underline{v}}{\partial t}+\rho\underline{v}\cdot\nabla\underline{v}\right)=-\nabla \mathrm{p}+\mu\nabla^2\underline{v}+\rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Convective terms

Ma ntum Balance

$$\frac{d\underline{\mathbf{P}}}{dt}+\sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i}=\sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i}+\underline{R}+M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial\underline{v}}{\partial t}+\rho\underline{v}\cdot\nabla\underline{v}\right)=-\nabla \mathrm{p}+\mu\nabla^2\underline{v}+\rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma Convective terms ntum Pressure forces

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma Convective terms ntum Pressure forces Viscous forces

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma[...]tum

Convective terms

Pressure forces

Viscous forces

Gravity force

$$\frac{d\underline{\boldsymbol{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\boldsymbol{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} \left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

Macroscopic Momentum Balance

Now we need to learn when and how to apply this

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Macroscopic Momentum Balance

$\underline{\mathbf{P}}$ = *fluid* momentum

$$\beta_{laminar} = 0.75$$
$$\beta_{turbulent} \sim 1$$

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$$\begin{pmatrix}\frac{d\mathbf{P}_x}{dt}\\ \frac{d\mathbf{P}_y}{dt}\\ \frac{d\mathbf{P}_z}{dt}\end{pmatrix}_{xyz} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\begin{pmatrix}\hat{v}_x\\ \hat{v}_y\\ \hat{v}_z\end{pmatrix}_{xyz}\right]_{A_i}$$

$$= \sum_{i=1}^{\#streams}\left[-pA\begin{pmatrix}\hat{n}_x\\ \hat{n}_y\\ \hat{n}_z\end{pmatrix}_{xyz}\right]_{A_i} + \begin{pmatrix}R_x\\ R_y\\ R_z\end{pmatrix}_{xyz} + M_{CV}\begin{pmatrix}g_x\\ g_y\\ g_z\end{pmatrix}_{xyz}$$

$\underline{R}$ = net force *on fluid* due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal of CV

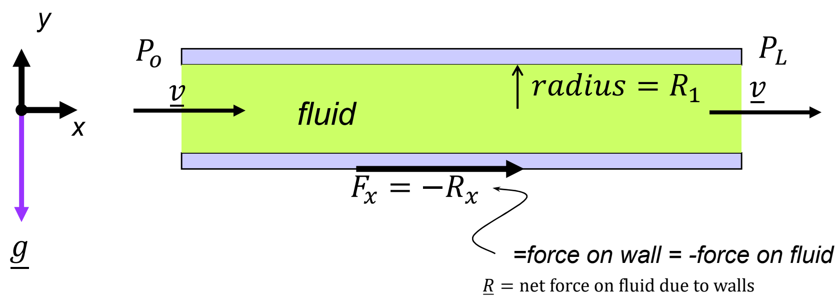
See inside front cover of Morrison, 2013

Macroscopic Momentum Balance Example:
Drag on the walls of a pipe

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

Macroscopic Momentum Balance Example:
Calculate the drag on the walls of a pipe

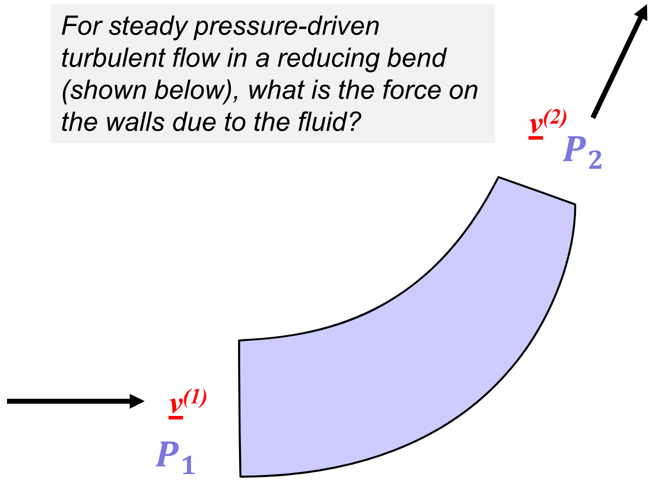
Assume:
- steady state
- turbulent
- neglect gravity



Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

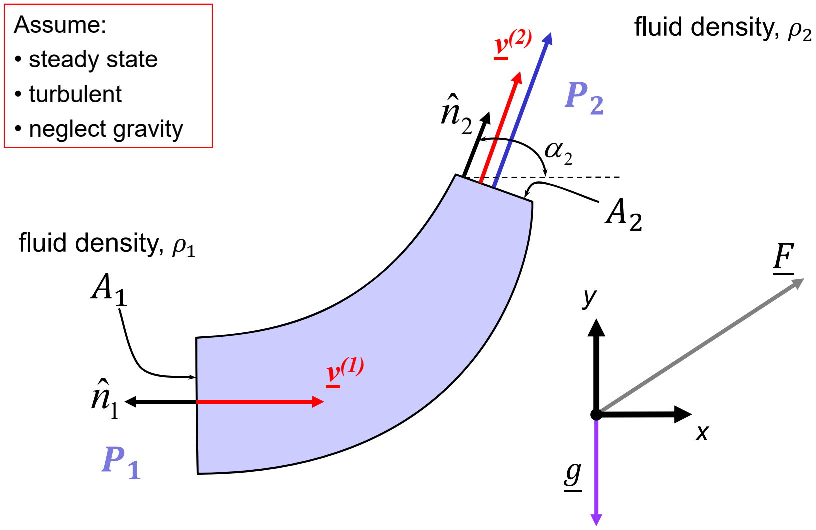
For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?



Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

Assume:
- steady state
- turbulent
- neglect gravity



Macroscopic Momentum Balance www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Types of Momentum Transfer

Microscopic		Macroscopic
convection (*momentum flows in*)	→	convection (*momentum flows in*)
pressure forces	→	pressure forces
viscous forces (*or viscous flux*)	→	wall forces (*due to viscosity*)
body forces (*gravity*)	→	body forces (*gravity*)

After calculating the flow field with microscopic balances you can calculate wall forces

With macroscopic balances you can often **calculate wall forces directly**

Macroscopic Momentum Balance www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Problem-Solving Procedure -

Steady State Macroscopic Momentum Problems

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

1. sketch system; choose CV on which you will perform balance
2. choose coordinate system
3. perform macroscopic mass balance

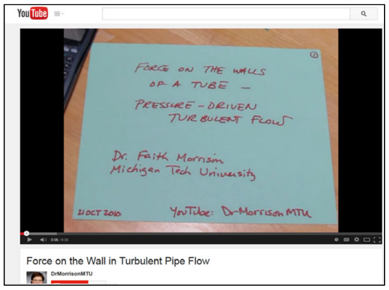
Consider angles carefully

4. perform macroscopic momentum balance (vector equation; forces are pressure, gravity, force on the wall; all forces ON the fluid in CV)
5. solve (usually for force on the wall)

Macroscopic Momentum Balance www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Solution to force on a reducing bend:

www.chem.mtu.edu/~fmorriso/cm310/reducing_bend.pdf



Dr. Morrison doing a Macro-Momentum Balance on YouTube:

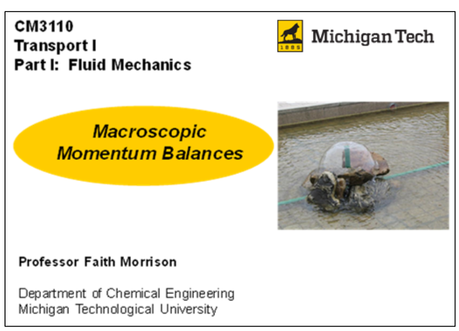
(DrMorrisonMTU)

www.youtube.com/watch?v=jXNkN7NMINM

(note that there is a sign error in the gravity term; sorry about that; gravity is negligible)

Many useful handouts:

www.chem.mtu.edu/~fmorriso/cm310/handouts.html



Done

(just need to practice; see HW4)

Bonus: When to use which?

Mechanical Energy Balance

$$\boxed{\frac{P_2 - P_1}{\rho} + \frac{\langle v\rangle_2^2 - \langle v\rangle_1^2}{2\alpha} + g(z_2 - z_1) + F_{21} = \frac{W_{s,on}}{\dot{m}}}$$

Macroscopic Momentum Balance

$$\boxed{\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}[-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}}$$

Microscopic Momentum Balance

$$\boxed{\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{P} + \mu\nabla^2\underline{v} + \rho\underline{g}}$$