Michigan Tech

CM3110
Transport I
Part I: Fluid Mechanics

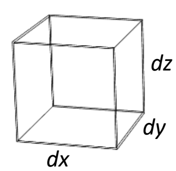
Microscopic Momentum Balance Equation (Navier-Stokes)

Professor Faith Morrison

Department of Chemical Engineering
Michigan Technological University

Microscopic Balances

We have been doing a microscopic *control volume* balance; these are specific to whatever problem we are solving.

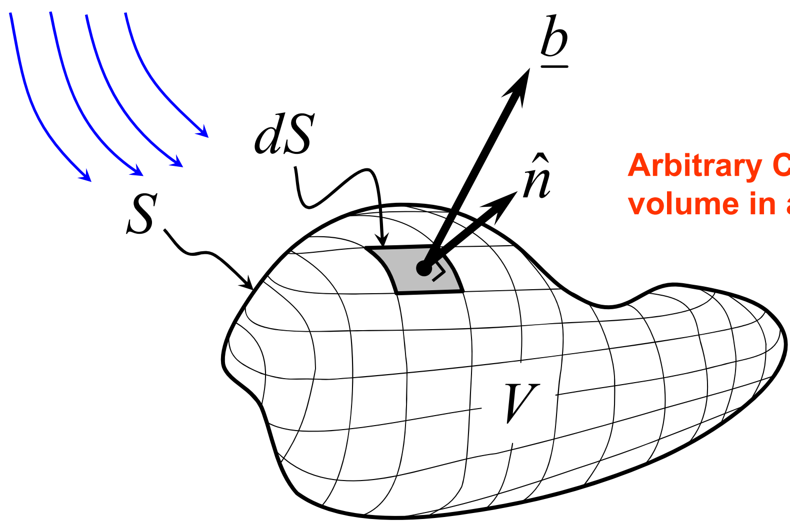


We seek equations for microscopic mass, momentum (and energy) balances that are *general*.

$\Rightarrow$

- equations must not depend on the choice of the control volume,
- equations must capture the appropriate balance

Microscopic Momentum Balance Equation (Navier-Stokes)



Arbitrary Control volume in a Flow

Mass Balance

On an arbitrary control volume:

$$\begin{pmatrix} \text{rate of} \\ \text{increase of mass} \\ \text{in control volume} \end{pmatrix} = \begin{pmatrix} \text{net flux of} \\ \text{mass into C.V.} \end{pmatrix}$$

(details in the book)

$$\iiint_V \frac{\partial}{\partial t}(\rho)\,dV = -\iiint_V \nabla \cdot (\rho \underline{v})\,dV$$

Rate of increase of mass — Net convection in

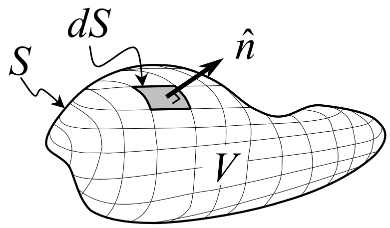
⇓

(just as we did with the individual control volume balance)

Microscopic mass balance for <u>any</u> flow

Continuity Equation

Microscopic **mass** balance written on an arbitrarily shaped control volume, *V*, enclosed by a surface, *S*



$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

Gibbs notation:

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho (\nabla \cdot \underline{v})$$

Microscopic mass balance is a scalar equation.

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Momentum Balance

On an arbitrary control volume:

(details in the book)

$$\begin{pmatrix} rate\ of\ increase \\ of\ momentum \\ in\ C.V. \end{pmatrix} = \begin{pmatrix} net\ flux\ of \\ momentum \\ into\ C.V. \end{pmatrix} + \begin{pmatrix} sum\ of\ forces \\ on\ C.V. \end{pmatrix}$$

$$\iiint_V \frac{\partial}{\partial t}(\rho \underline{v}) dV = -\iiint_V \nabla \cdot (\rho \underline{v}\,\underline{v}) dV + \iiint_V \rho \underline{g}\, dV - \iiint_V \nabla \cdot \underline{\underline{\tilde{\Pi}}}\, dV$$

Rate of increase of momentum | Net convection in | Force due to gravity | Viscous forces and pressure forces

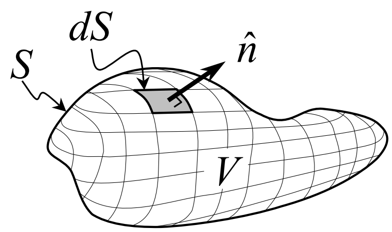
$$\Downarrow$$

(just as we did with the individual control volume balance)

Microscopic momentum balance for any flow

Microscopic Momentum Balance Equation (Navier-Stokes)

Equation of Motion



Microscopic **momentum** balance written on an arbitrarily shaped control volume, *V*, enclosed by a surface, *S*

Gibbs notation: $$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \nabla\cdot\underline{\underline{\tilde{\tau}}} + \rho\underline{g}$$ **general fluid**

Gibbs notation: $$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$$ **Newtonian fluid**

Navier-Stokes Equation

Microscopic momentum balance is a vector equation.

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Continuity Equation
(And Non-Newtonian Equation) on the FRONT

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

Continuity Equation, Cartesian coordinates

$$\frac{\partial\rho}{\partial t} + \left(v_x\frac{\partial\rho}{\partial x} + v_y\frac{\partial\rho}{\partial y} + v_z\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

$$\frac{\partial\rho}{\partial t} + \underline{v}\cdot\nabla\rho = -\rho(\nabla\cdot\underline{v})$$

Continuity Equation, spherical coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial\phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial\tilde{\tau}_{xx}}{\partial x} + \frac{\partial\tilde{\tau}_{yx}}{\partial y} + \frac{\partial\tilde{\tau}_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial\tilde{\tau}_{xy}}{\partial x} + \frac{\partial\tilde{\tau}_{yy}}{\partial y} + \frac{\partial\tilde{\tau}_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial\tilde{\tau}_{xz}}{\partial x} + \frac{\partial\tilde{\tau}_{yz}}{\partial y} + \frac{\partial\tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z$$

The one with "$\tilde{\tau}$" is for non-Newtonian fluids

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www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

Navier-Stokes (Newtonian Fluids Only) is on the BACK:

There are no "$\tilde{\tau}$"`s on this side

$$\rho\left(\frac{\partial \underline{v}}{\partial t}+\underline{v}\cdot\nabla\underline{v}\right)=-\nabla p+\mu\nabla^2\underline{v}+\rho\underline{g}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

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Problem-Solving Procedure – solving for velocity and stress fields

amended: when using the microscopic balances

1. sketch system
2. choose coordinate system
3. simplify the continuity equation (mass balance)
4. simplify the 3 components of the equation of motion (momentum balance) (note that for a Newtonian fluid, the equation of motion is the Navier-Stokes equation)
5. solve the <u>differential</u> equations for velocity and pressure (if applicable)
6. apply boundary conditions
7. calculate any engineering values of interest (flow rate, average velocity, force on wall)

$$\underline{\underline{\tilde{\tau}}}=\mu\left(\nabla\underline{v}+(\nabla\underline{v})^T\right)$$

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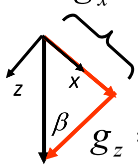
EXAMPLE I: Flow of a Newtonian fluid down an inclined plane

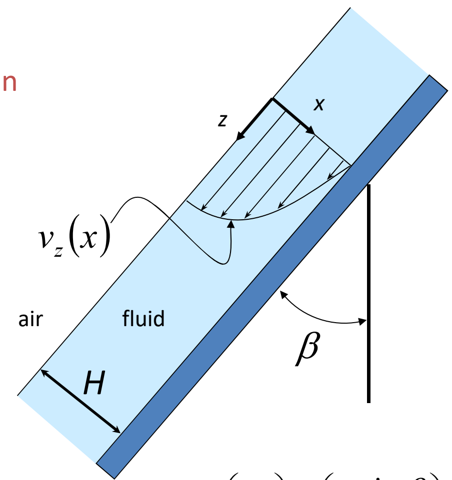
Revisited

$$g_x = g \sin\beta$$



$$g_z = g \cos\beta$$

$\underline{g}$



$$\underline{g} = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}$$

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EXAMPLE I: Flow of a Newtonian fluid down an inclined plane

Revisited

(see hand notes)

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Microscopic Momentum Balance Equation (Navier-Stokes)

As with balance we performed with a control volume we selected, we made modelling assumptions along the way that we can collect and associate with the final result:

Model Assumptions: (laminar flow down an incline, Newtonian)

1. no velocity in the *x*- or *y*-directions (laminar flow) $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz}$
2. well developed flow
3. no edge effects in *y*-direction (width) $v_z \neq f(y)$
4. constant density
5. steady state
6. Newtonian fluid
7. no shear stress at interface
8. no slip at wall

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Calculate the steady state velocity field of an incompressible, Newtonian fluid flowing in a long pipe of circular cross section. The pressure at the entrance of the section of interest is P_0 and the pressure a distance L downstream is P_L.

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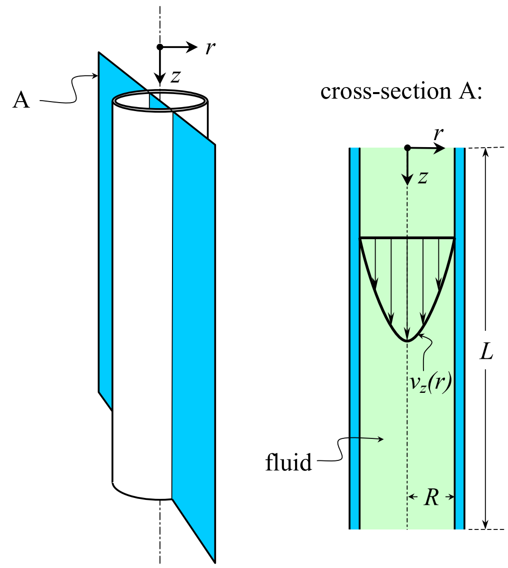

EXAMPLE II:
Pressure-driven flow of a Newtonian fluid in a tube:
(Poiseuille flow)

- steady state
- constant ρ
- well developed
- long tube
- pressure p_o at top
- pressure p_L at bottom

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The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial (\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial (\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial \tilde{\tau}_{xx}}{\partial x} + \frac{\partial \tilde{\tau}_{yx}}{\partial y} + \frac{\partial \tilde{\tau}_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial \tilde{\tau}_{xy}}{\partial x} + \frac{\partial \tilde{\tau}_{yy}}{\partial y} + \frac{\partial \tilde{\tau}_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial \tilde{\tau}_{xz}}{\partial x} + \frac{\partial \tilde{\tau}_{yz}}{\partial y} + \frac{\partial \tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z$$

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www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

Navier-Stokes:

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right)+\rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial P}{\partial \theta}+\mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(rv_\theta)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z}+\mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

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Poiseuille flow of a Newtonian fluid:

See hand notes

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List of Common Integrals

Prof. Faith Morrison, Michigan Technological University
CM3110 Transport Processes I
CM4650 Polymer Rheology

$$\int du = u + C$$

$$\int u\, du = \frac{u^2}{2} + C$$

$$\int u^n\, du = \frac{u^{n+1}}{n+1} + C$$

$$\int \frac{1}{u}\, du = \ln|u| + C$$

$$\int \frac{1}{u^2}\, du = -\frac{1}{u} + C$$

$$\int e^u\, du = e^u + C$$

$$\int \ln u\, du = u \ln u - u + C$$

$$\int a^u du = \left(\frac{1}{\ln a}\right) a^u + C$$

$$\int \sin u\, du = -\cos u + C$$

$$\int \cos u\, du = \sin u + C$$

$$\int u\, e^u\, du = e^u(u-1) + C$$

$$\int u^2 e^u\, du = e^x(x^2 - 2x + 2) + C$$

$$\int e^u \sin u\, du = \frac{e^u(\sin u - \cos u)}{2} + C$$

$$\int e^u \cos u\, du = \frac{e^u(\sin u + \cos u)}{2} + C$$

$$\int e^{au} \sin bu = \frac{e^{au}}{(a^2+b^2)}(a \sin bu - b \cos bu) + C$$

$$\int e^{au} \cos bu = \frac{e^{au}}{(a^2+b^2)}(b \sin bu + a \cos bu) + C$$

Integration by parts:

$$\int u\, dv = uv - \int v\, du$$

Note: Many more integrals can be solved with these formulas by using *substitution*. For example, evaluate I:

$$I \equiv \int (1-2x)^2 dx$$

Let $u = (1-2x)$; then

$$\frac{du}{dx} = -2, \qquad du = -2\, dx$$

Substituting into the integral, we obtain

$$I = -\frac{1}{2}\int u^2 du = \left(-\frac{1}{2}\right)\frac{u^3}{3} + C$$

Substituting back (eliminating u) gives the answer:

$$I = -\frac{1}{6}(1-2x)^3 + C$$

29 April 2014

List of Common Integrals

www.chem.mtu.edu/~fmorriso/cm310/2014CommonIntegrals.pdf

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Common Boundary Conditions in Fluid Mechanics

1. *No-slip at the wall.* This boundary condition says that the fluid in contact with a wall will have the same velocity as the velocity of the wall. Often the walls are not moving, so the fluid velocity is zero. In drag flows like the previous example, the velocity of the wall is finite and the fluid velocity is equal to the wall velocity.

$$v_p\big|_{\text{at the boundary}} = V_{\text{wall}} \qquad (1)$$

2. *Symmetry.* In some flows there is a plane of symmetry. Since the velocity field is the same on either side of the plane of symmetry, the velocity must go through a minimum or a maximum at the plane of symmetry. Thus, the boundary condition to use is that the first derivative of the velocity is zero at the plane of symmetry.

$$\left.\frac{\partial v_p}{\partial x_m}\right|_{\text{at the boundary}} = 0 \qquad (2)$$

3. *Stress continuity.* When a fluid forms one of the boundaries of the flow, the stress is continuous from one fluid to another. Thus for a viscous fluid in contact with an inviscid (zero or very low viscosity fluid), this means that at the boundary, the stress in the viscous fluid is the same as the stress in the inviscid fluid. Since the inviscid fluid can support no shear stress (zero viscosity) this means that the stress is zero at this interface. The boundary condition between a fluid such as a polymer and air, for example, would be that the shear stress in the polymer at the interface would be zero.

$$\tau_{jk}\big|_{\text{at the boundary}} = 0 \qquad (3)$$

Alternatively if two viscous fluids meet and form a flow boundary, this same boundary condition would require that the stress in one fluid equal the stress in the other at the boundary.

$$\tau_{jk}(\text{fluid 1})\big|_{\text{at the boundary}} = \tau_{jk}(\text{fluid 2})\big|_{\text{at the boundary}} \qquad (4)$$

4. *Velocity continuity.* When a fluid forms one of the boundaries of the flow as described above, the velocity is also continuous from one fluid to another.

$$v_p(\text{fluid 1})\big|_{\text{at the boundary}} = v_p(\text{fluid 2})\big|_{\text{at the boundary}} \qquad (5)$$

List of Common Boundary Conditions

http://www.chem.mtu.edu/~fmorriso/cm310/bc.pdf

Poiseuille flow of a Newtonian fluid:

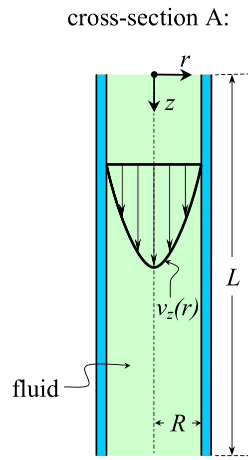
What is the force on the walls in this flow?

$$\int \left(\frac{\text{force}}{\text{area}}\right)(area)$$

Total wetted area

$$= \int (\tilde{\tau}_{??})(area)$$

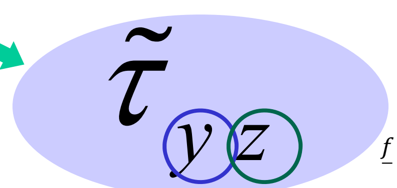
Inside surface of tube

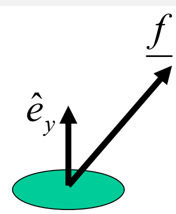


$$\tilde{\tau}_{yz} = \frac{force}{area} = \frac{kg\ m/s^2}{area} = \frac{(kg)(m/s)}{(s)(area)}$$

Momentum Flux

9 stresses at a point in space





$\underline{f} = A(\tilde{\tau}_{yz}\hat{e}_x + \tilde{\tau}_{yy}\hat{e}_y + \tilde{\tau}_{yz}\hat{e}_z)$

stress on a *y*-surface → in the *z*-direction

A surface whose unit normal is in the y-direction

$-\tilde{\tau}_{yz}$ = flux of *z*-momentum ← in the *y*-direction

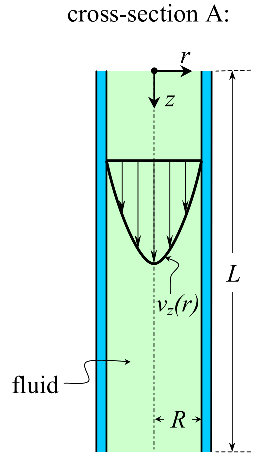
(See discussion of sign convention of stress; this is the tension-positive convention, $\tilde{\tau}_{ij}$)

Poiseuille flow of a Newtonian fluid:

What is the **shear stress** in this flow?

$$\tilde{\tau}_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Stress on an r surface
in the z direction



Poiseuille flow of a Newtonian fluid:

Force on the walls:

See hand notes

Poiseuille flow of a Newtonian fluid:

$$p(z) = \frac{-(p_0 - p_L)}{L} z + p_o$$

$$\tilde{\tau}_{rz}(r) = \frac{-(\rho g L + p_o - p_L)}{2L} r$$

$$v(r) = \frac{R^2(\rho g L + p_o - p_L)}{4\mu L}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

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Engineering Quantities of Interest

(tube flow)

average velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi}\int_0^R v_z r dr d\theta}{\int_0^{2\pi}\int_0^R r dr d\theta}$$

volumetric flow rate

$$Q\rangle = \int_0^{2\pi}\int_0^R v_z r dr d\theta = \pi R^2 \langle v_z \rangle$$

z-component of force on the wall

$$F_z = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz}\Big|_{r=R} R d\theta dz$$

Must work these out for each problem in the coordinate system in use; see inside back cover of book.

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Engineering Quantities of Interest

(any flow)

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component of force on the wall

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}} \right) \right]_{surface} dS$$

For more complex flows, we use the ***Gibbs notation*** *versions (will discuss soon).*

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Poiseuille flow of a Newtonian fluid:

Volumetric Flow Rate:

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

Let's try

See hand notes

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Poiseuille flow of a Newtonian fluid:

$$Q = \int_0^{2\pi}\int_0^R v_z(r) r dr d\theta$$

$$= \frac{R^2(\rho g L + p_o - p_L)}{4\mu L}\int_0^{2\pi}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr d\theta$$

$$Q = \frac{\pi R^4(\rho g L + p_0 - p_L)}{8\mu L}$$

Hagen-Poiseuille Equation**

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Poiseuille flow of a Newtonian fluid:

$$v_{av} = \int_0^{2\pi}\int_0^R v_z(r) r dr d\theta$$

$$= \frac{R^2(L\rho g + P_o - P_L)}{4\mu L}\int_0^{2\pi}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr d\theta$$

$$= \frac{R^2(L\rho g + P_o - P_L)}{8\mu L}$$

$$= \frac{v_{z,\max}}{2}$$

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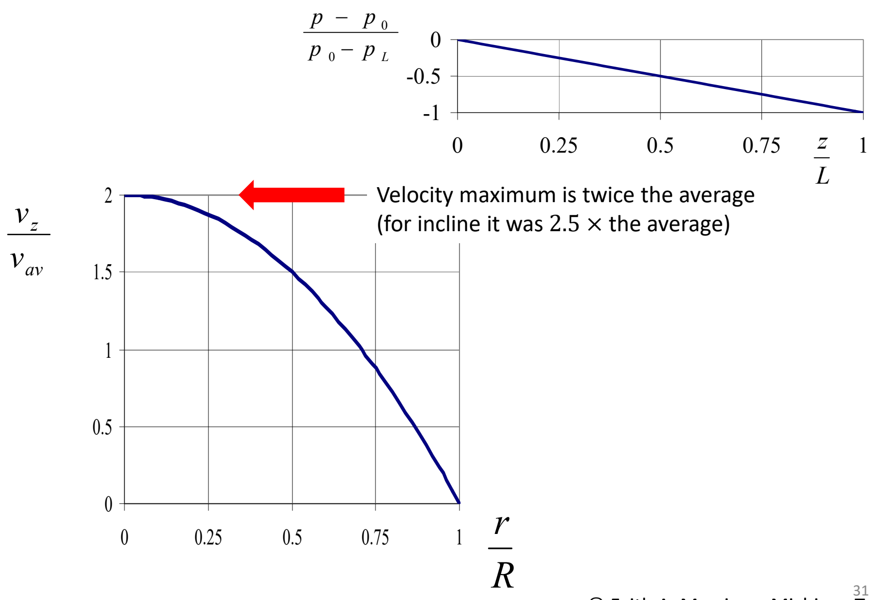
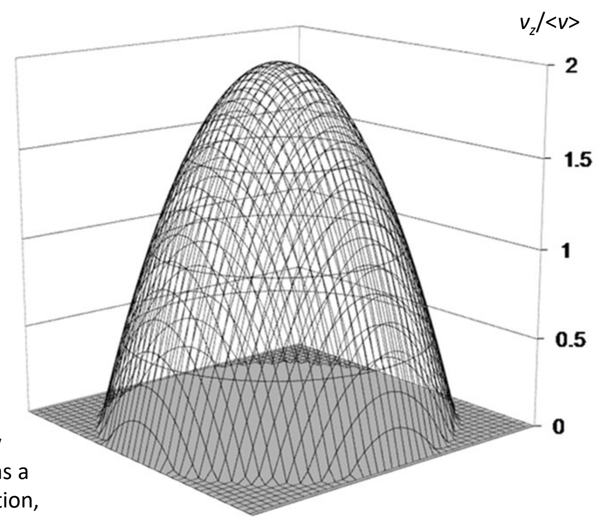
Poiseuille flow of a Newtonian fluid:



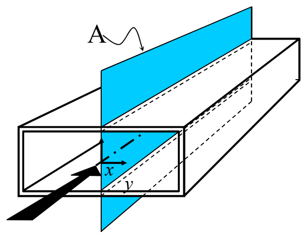
Velocity maximum is twice the average (for incline it was 2.5 × the average)

EXAMPLE II:
Pressure-driven flow of a Newtonian fluid in a tube:
Poiseuille flow



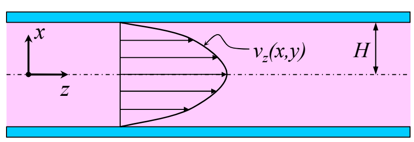
Bullet shaped; flow down an incline was a parabola cross-section, but a sheet in 3d.

Example III: Pressure-driven flow of a Newtonian fluid in a rectangular duct: **Poiseuille flow**



What is the steady state velocity profile for laminar flow of an incompressible Newtonian fluid flowing down a long duct of rectangular cross section? The duct height is $2H$ and the width is $2W$. The pressure at an upstream position is p_0 and a distance L downstream the pressure is p_L. What is the force on the walls?

cross-section A:



See hand notes

(Example 7.11, p549)

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Michigan Tech

Can this modeling method work for complex flows?

Answer: *yes.* (with some qualifiers)

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