10 + 12 Sept 2019 FAM

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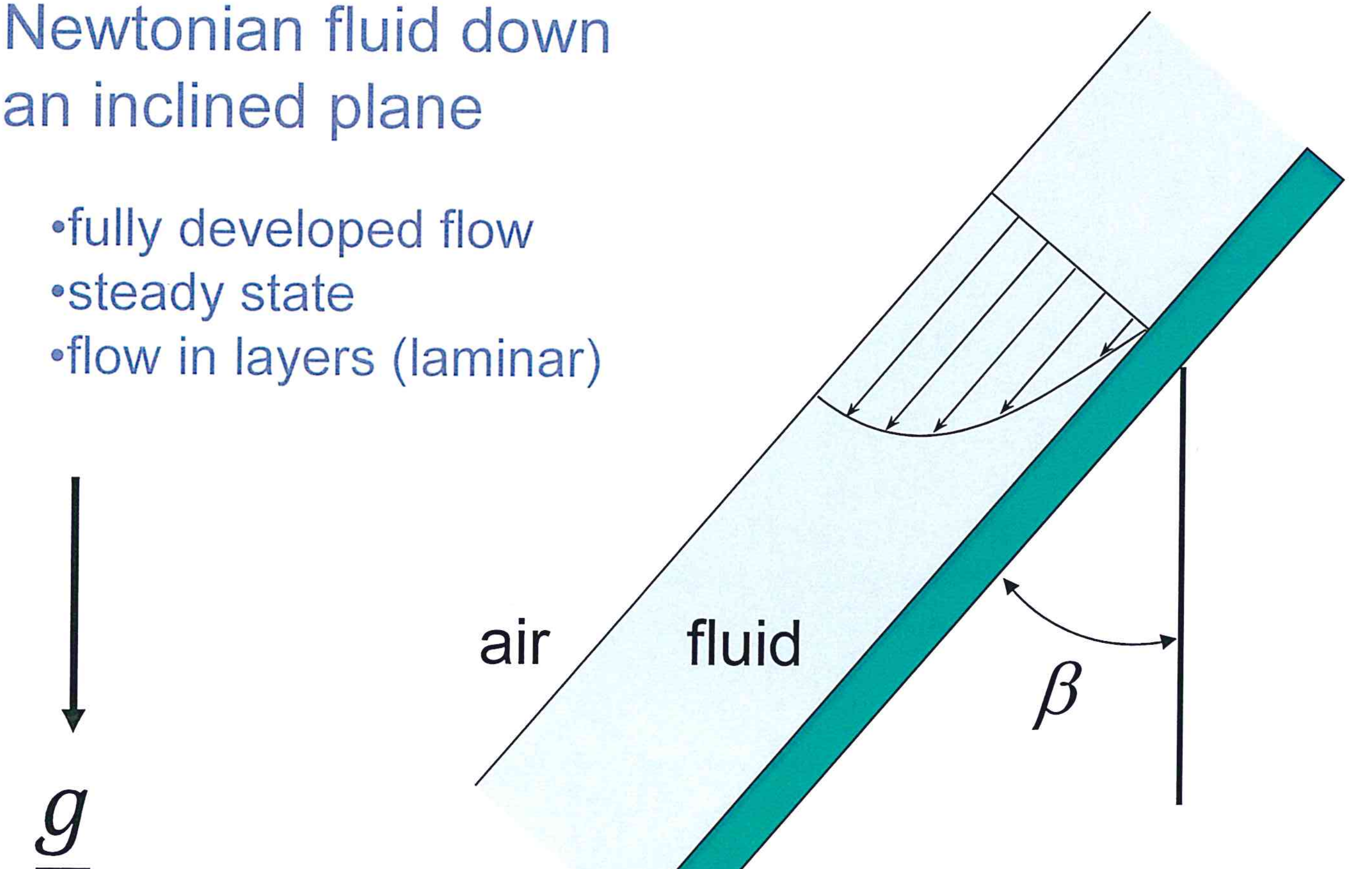
EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

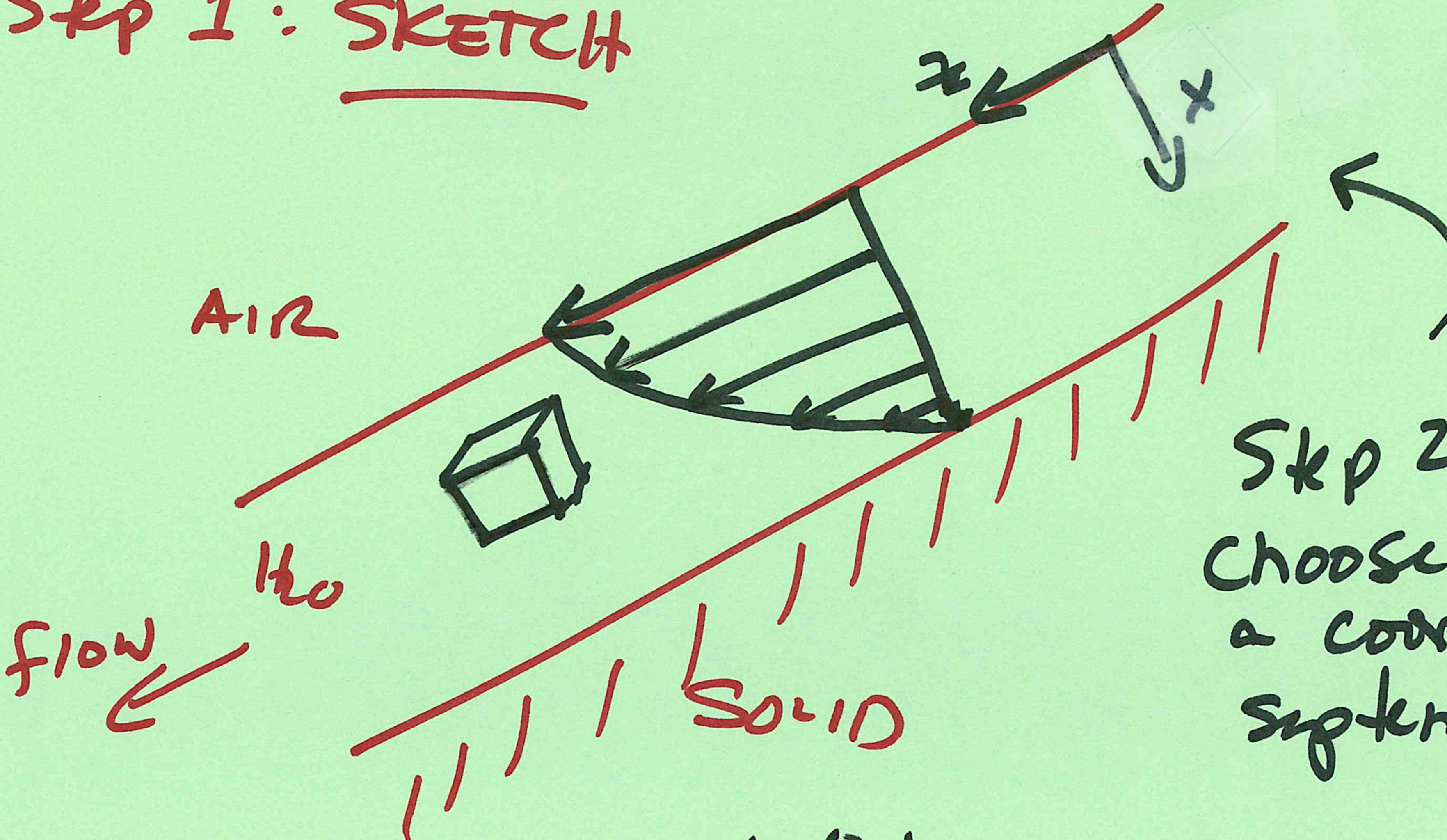
- fully developed flow
- steady state
- flow in layers (laminar)



EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

What is the velocity field in the steady flow of water down a slope that is wide and long. The fluid properties are constant, and the flow is driven by gravity. The flow is slow so that no waves are formed. What is the force on the surface due to the water flow? What is the flow rate?

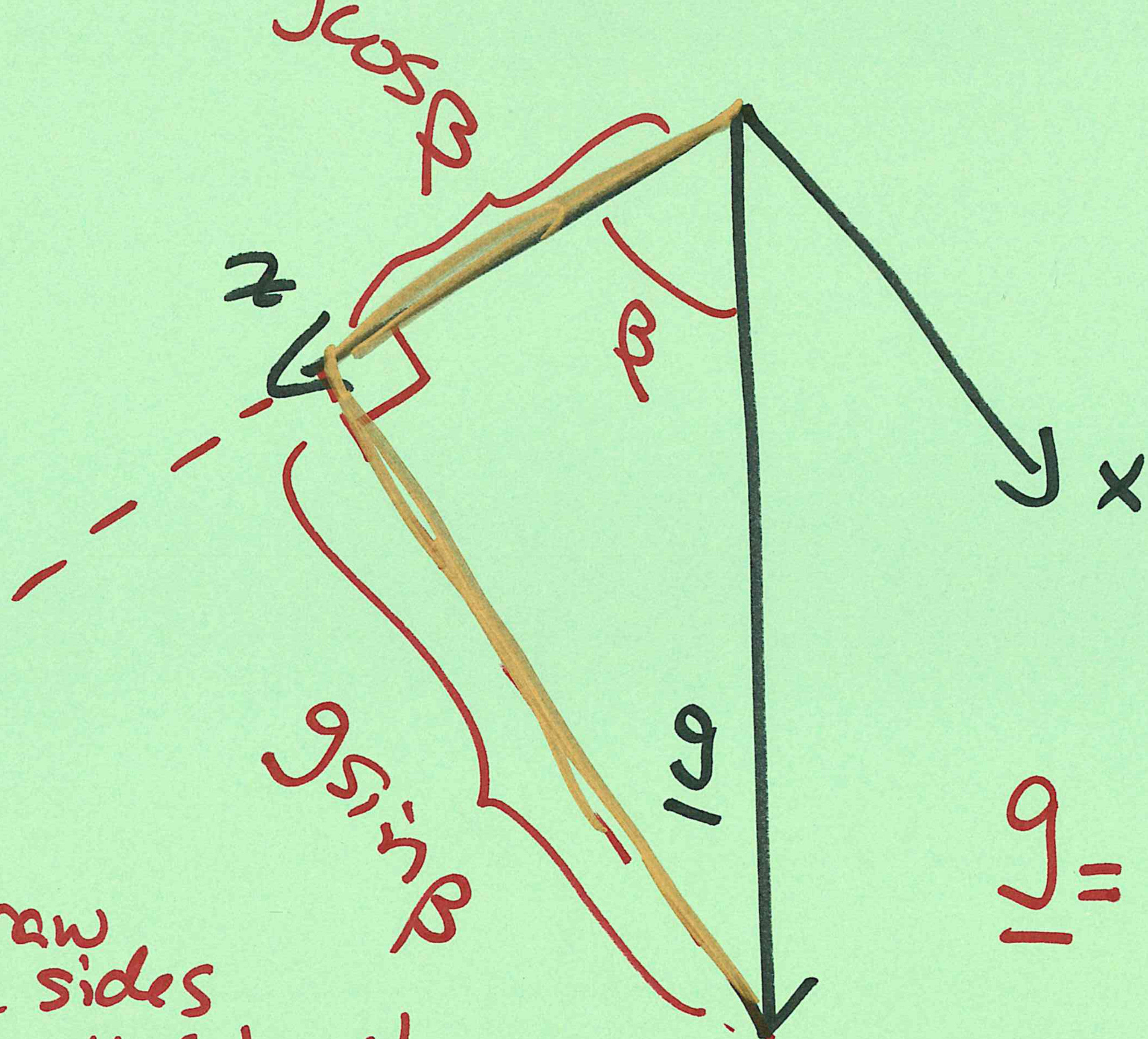
Step 1 : SKETCH



Step 2:
Choose
a coord.
system

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz} = v_z \hat{e}_z$$

DECOMPOSE $\underline{g}$ into our coordinate system:



Draw the sides parallel to the coordinate axes.

$$\underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz}$$

What are we trying to calculate:

<u>VELOCITY FIELD</u>

$$\frac{m}{s} \; [=] \; \underline{v} = \begin{pmatrix} \cancel{v_x} \\ \cancel{v_y} \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} v_x(x,y,\cancel{z}) \\ v_y(x,y,\cancel{z}) \\ v_z(x,\cancel{y},\cancel{z}) \end{pmatrix}_{xyz}$$

- vector
- a function of position

Choose a c.v. that allows you to calc $\underline{v}(x,y,z)$.

CONTROL VOLUME BALANCES

MASS

$$\begin{pmatrix}\text{net mass in}\end{pmatrix} = \begin{pmatrix}\text{rate of accumulation of mass}\end{pmatrix}$$

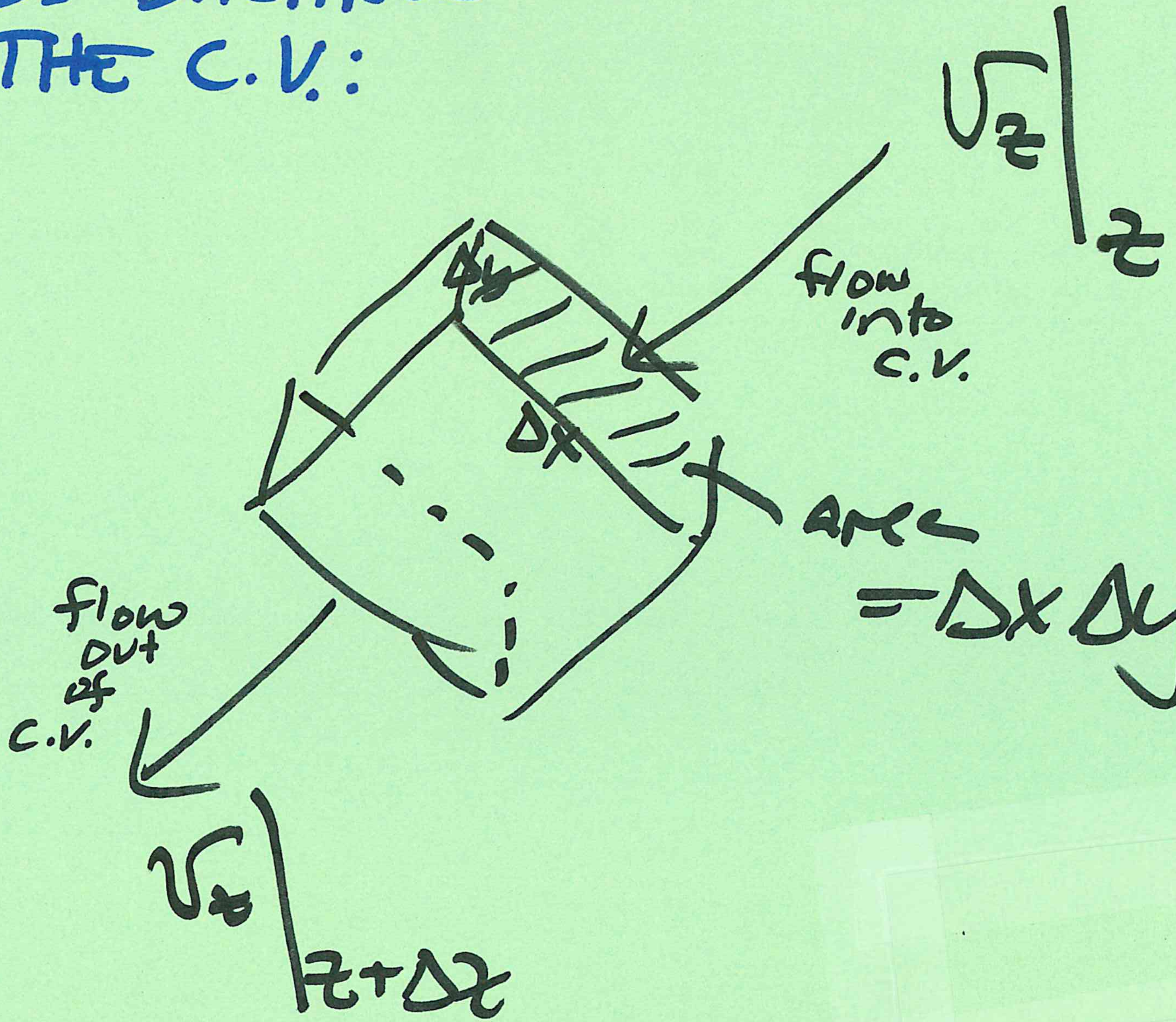
MOMENTUM

$$\begin{pmatrix}\text{sum of forces on C.V.}\end{pmatrix} = \underbrace{\begin{pmatrix}\text{momentum in C.V.}\end{pmatrix} - \begin{pmatrix}\text{momentum out C.V}\end{pmatrix}}_{\text{"convective terms"}}$$

$$= \begin{pmatrix}\text{rate of accumulation of momentum in C.V.}\end{pmatrix}$$

"convective terms"

MASS BALANCE ON THE C.V.:



Recall: $\langle v \rangle = \frac{Q}{\text{AREA}}$

MASS INTO C.V.

$$= \rho \left(\frac{\text{mass}}{\cancel{\text{volume}}}\right) \left(\frac{\cancel{\text{volume}}}{\text{time}}\right) = \frac{\text{mass}}{\text{time}}$$

MASS in C.V. $= \rho \, v_z\big|_z (\Delta x \, \Delta y)$

MASS out $= \rho \, v_z\big|_{z+\Delta z} (\Delta x \, \Delta y)$

MASS in = MASS out

$$\cancel{\rho}\, v_z\big|_z\, \cancel{\Delta x}\, \cancel{\Delta y} = \cancel{\rho}\, v_z\big|_{z+\Delta z}\, \cancel{\Delta x}\, \cancel{\Delta y}$$

$$v_z\big|_z = v_z\big|_{z+\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{v_z\big|_{z+\Delta z} - v_z\big|_z}{\Delta z} = \boxed{\frac{dv_z}{dz} = 0}$$

MOMENTUM BALANCE

CONVECTIVE TERMS

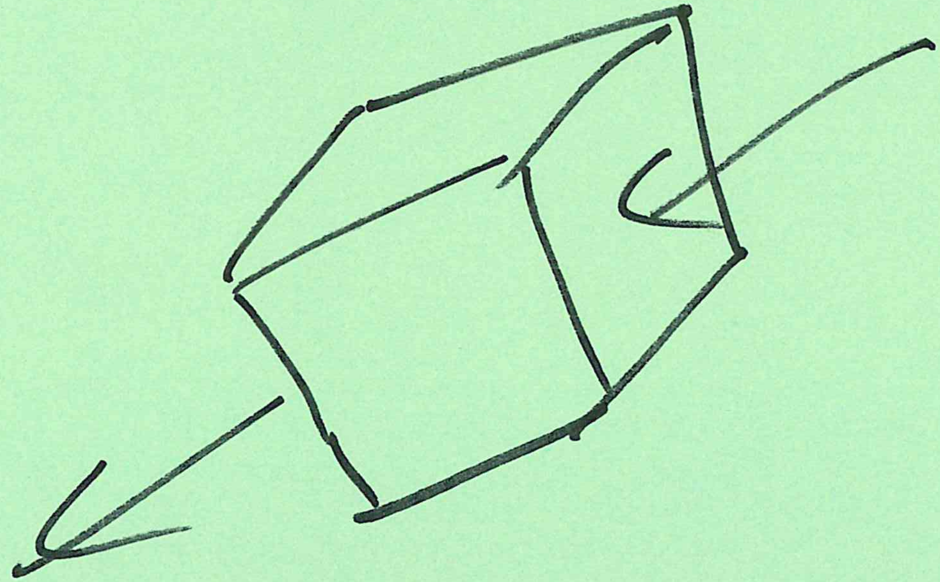
momentum

momentum ~~into~~ out C.V. $= \left(\frac{\text{MASS}}{\text{time}}\right)(\text{velocity})$

$$= \left(\rho \, v_z\Big|_{\cancel{z}\ z+\Delta z} \Delta x \Delta y\right)\left(v_z\Big|_{\cancel{z}\ z+\Delta z}\right)$$

$$= \rho \left(v_z\Big|_{\cancel{z}\ z+\Delta z}\right)^2 \Delta x \Delta y$$

FORCES – GRAVITY



Force of gravity on the c.V = (mass)(acceleration)

$$= \rho\,(\text{volume})\;\underline{g}$$

$$= \rho\,(\Delta x \Delta y \Delta z)\;\underline{g}$$

From before:

$$\underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz}$$

(in our coord system)

the z-momentum due to gravity $= \rho g \cos\beta \, \Delta x \Delta y \Delta z$

FORCES - VISCOUS

Force on top $-\tau_{xz}\Big|_{x} \Delta z \Delta y$

stress on an x surface in z-dir

$\left(\frac{\text{FORCE}}{\text{Area}}\right)(\text{AREA})$

(negative sign is to match stress sign convention of our text)

Force on bottom $-\tau_{xz}\Big|_{x+\Delta x} \Delta z \Delta y$

Pull together the momentum balance:

$$\sum \underset{\text{C.V}}{\text{FORCES on}} = \underset{\text{in}}{\text{Momentum}} - \underset{\text{out}}{\text{momentum}}$$

$$(\text{gravity}) + (\text{viscous}) = \quad '' \qquad\qquad '' $$

$$\rho \Delta x \cancel{\Delta y} \cancel{\Delta z} \cos\beta g + \left(-\tilde{\tau}_{xz}\Big|_{x} \cancel{\Delta z} \cancel{\Delta y} \right)$$

$$- \left(-\tilde{\tau}_{xz}\Big|_{x+\Delta x} \cancel{\Delta z} \cancel{\Delta y} \right)$$

$$= \cancel{\rho \left(v_z\Big|_{z} \right)^2 \Delta x \Delta y} - \cancel{\rho \left(v_z\Big|_{z+\Delta z} \right)^2 \Delta x \Delta y}$$

$$\rho g \cos\beta \, \Delta x = \left(-\tilde{\tau}_{xz}\right)\Big|_{x+\Delta x} - \left(-\tilde{\tau}_{xz}\right)\Big|_{x}$$

Recall:
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \equiv \frac{df}{dx}$$

$$\lim_{\Delta x \to 0} \frac{\left(-\tilde{\tau}_{xz}\right)\Big|_{x+\Delta x} - \left(-\tilde{\tau}_{xz}\Big|_{x}\right)}{\Delta x} = \rho g \cos\beta$$

$$\frac{d\left(\tilde{\tau}_{xz}\right)}{dx} = -\rho g \cos\beta$$

Let $\phi = \tilde{\tau}_{xz}$

Let $a = -\rho g \cos\beta$

$$\frac{d\phi}{dx} = a$$

$$\phi = ax + C_1$$

$$\tilde{\tau}_{xz} = (-\rho g \cos\beta)x + C_1$$

We use Newton's Law of Viscosity to relate $\tilde{\tau}_{xz}$ to the velocity field.

Newton's Law of Viscosity $\Rightarrow$ $\tilde{\tau}_{xz} = \mu \dfrac{dV_z}{dx}$

Equate:

$$\mu \frac{dV_z}{dx} = -(\rho g \cos\beta)x + C_1$$

$$\frac{dV_z}{dx} = -\frac{\rho g \cos\beta}{\mu} x + \frac{C_1}{\mu}$$

Integrate.

$$\boxed{V_z = -\frac{\rho g \cos\beta}{\mu} \frac{x^2}{2} + \frac{C_1}{\mu} x + C_2}$$

WE NEED TWO BOUNDARY CONDITIONS

BC1:

no slip at wall ("at the" wall) speed = speed of wall

BC: $x = H$ ($v_z = 0$)
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$$0 = \frac{\overset{-\rho g \cos\beta}{\overset{\shortparallel}{\alpha}}}{2\mu} H^2 + \frac{c_1}{\mu} H + c_2$$

★ one eqn, two unknowns

SECOND B.C.:

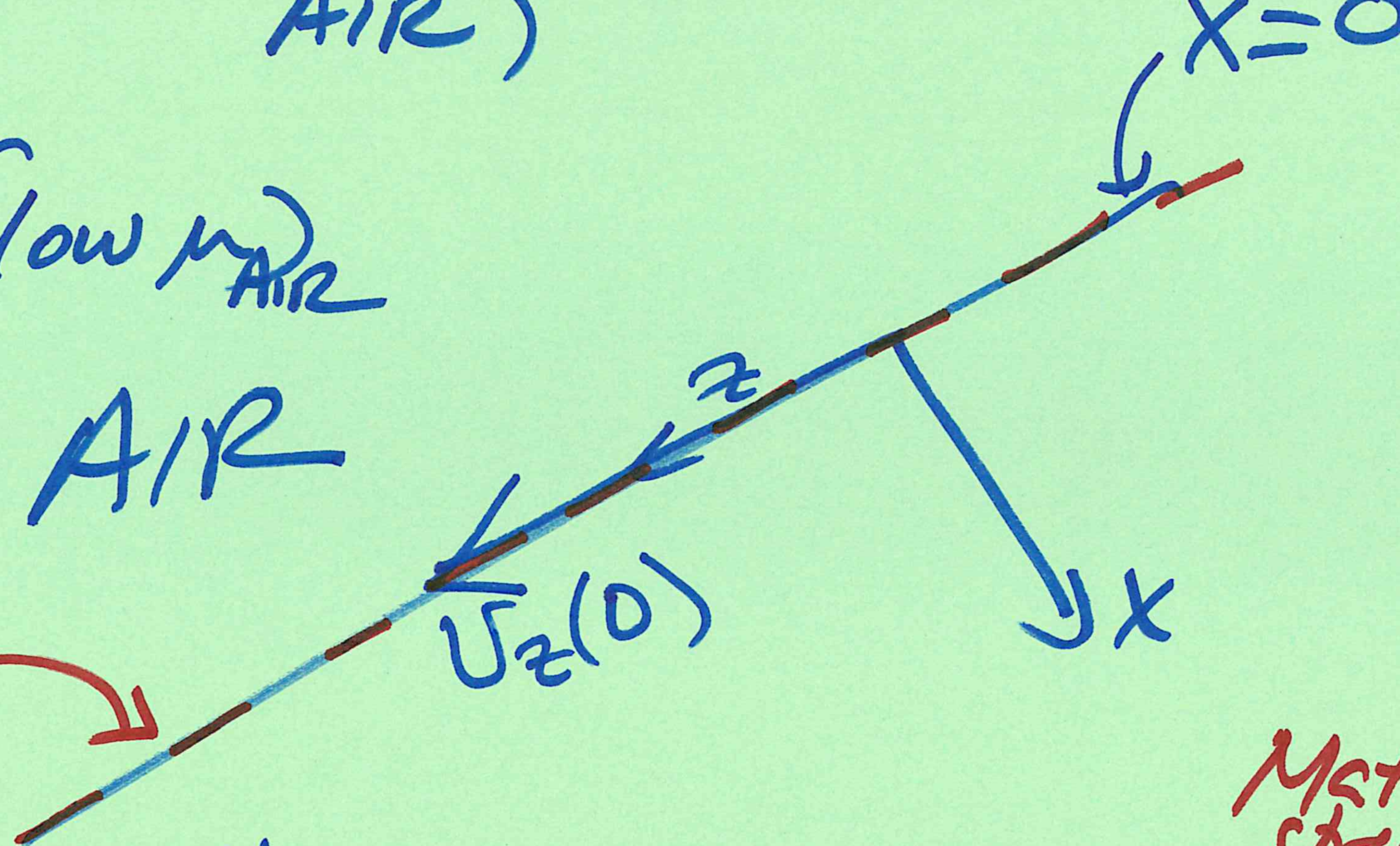
BC_2: ???

$\underbrace{x = 0}_{\text{location}}$ $\underbrace{???}_{\text{condition?}}$

At the "free" surface (in contact w/ AIR)

(low μ_{AIR})

AIR



inter-face

H_2O

μ_{WATER}

Match stresses at boundary

BC 2:

$$x=0 \qquad \tau_{xz}^{AIR} = \tau_{xz}^{H_2O}$$

Newton's Law:

$$0 \cong \mu_{AIR} \left.\frac{dV_z}{dx}\right|_{x=0}^{\text{IN AIR}} = \mu_{H_2O} \left.\frac{dV_z}{dx}\right|_{x=0}^{\text{IN } H_2O}$$

$$\mu_{H_2O} >> \mu_{air}$$

$$\Rightarrow \boxed{\left.\frac{dV_z}{dx}\right|_{x=0} \approx 0}$$

With these two boundary conditions we are now able to solve 2 eqns, 2 unknowns for C_1 & C_2 to obtain the final answer for $v_z(x)$.

BC1:
$$0 = -\frac{\rho g \cos\beta H^2}{2\mu} + \frac{C_1 H}{\mu} + C_2$$

BC2:
$$0 = \left.\frac{dv_z}{dx}\right|_{x=0} = -\frac{\rho g \cos\beta}{\mu}(0) + \frac{C_1}{\mu}$$

$$\Rightarrow \boxed{C_1 = 0}$$

$$\Rightarrow \boxed{C_2 = \frac{\rho g \cos\beta H^2}{2\mu}}$$